

Some topics related to metrics and norms, including ultrametrics and ultranorms, 3

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Abstract

Some basic geometric properties related to connectedness and topological dimension 0 are discussed, especially in connection with the ultrametric version of the triangle inequality.

Contents

I	Semimetrics and uniform structures	3
1	Semimetrics	3
2	Relations	5
3	Uniform structures	6
4	The associated topology	8
5	Closure and regularity	10
6	Symmetry and interior	12
7	Compactness	13
8	Semi-ultrametrics	15
9	Uniform continuity	16
10	Compactness, continued	18
11	Totally bounded sets	19
12	Induced uniform structures	21
13	Induced uniform structures, continued	22

14 Sub-bases	23
15 Cartesian products	24
16 Cartesian products, continued	26
17 Compatible semimetrics	27
18 Collections of semimetrics	28
19 Sequences of semimetrics	30
20 Collections of semi-ultrametrics	31
21 q -Semimetrics	33
22 q -Absolute value functions	34
23 q -Seminorms	35
 II Connectedness and dimension 0	 37
24 Connected sets	37
25 U -Separated sets	38
26 Uniformly separated sets	40
27 Equivalence relations	41
28 U -Chains	43
29 Chain connectedness	45
30 Chain connectedness, continued	46
31 Connectedness and closure	48
32 Two related uniformities	49
33 Topological dimension 0	51
 III Topological groups	 52
34 Basic notions	52
35 Associated uniformities	54

36 Some additional properties	55
37 Translation-invariant relations	56
38 Translation-invariant semimetrics	58
39 Translation-invariance and topology	59
40 Translation-invariant relations, continued	60
41 Uniform separation	62
42 Open subgroups	64
43 Commutative groups	65
References	67

Part I

Semimetrics and uniform structures

1 Semimetrics

Let X be a set. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *semimetric* on X if it satisfies the following three conditions.

First,

$$(1.1) \quad d(x, x) = 0$$

for every $x \in X$. Second, $d(x, y)$ should be symmetric in x and y , so that

$$(1.2) \quad d(x, y) = d(y, x)$$

for every $x, y \in X$. Third, $d(\cdot, \cdot)$ should satisfy the *triangle inequality*, which is to say that

$$(1.3) \quad d(x, z) \leq d(x, y) + d(y, z)$$

for every $x, y, z \in X$. If $d(\cdot, \cdot)$ also has the property that

$$(1.4) \quad d(x, y) > 0$$

for every $x, y \in X$ with $x \neq y$, then $d(\cdot, \cdot)$ is said to be a *metric* on X . The *discrete metric* on X is defined by putting $d(x, y)$ equal to 1 when $x \neq y$ and to 0 when $x = y$, and it is easy to see that this defines a metric on X .

Let $d(\cdot, \cdot)$ be any semimetric on X . The *open ball* in X centered at a point $x \in X$ with radius $r > 0$ associated to $d(\cdot, \cdot)$ is defined as usual by

$$(1.5) \quad B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \geq 0$ associated to $d(\cdot, \cdot)$ is defined by

$$(1.6) \quad \overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

If $y \in B(x, r)$, then

$$(1.7) \quad t = r - d(x, y)$$

is strictly positive, and one can check that

$$(1.8) \quad B(y, t) \subseteq B(x, r),$$

using the triangle inequality. In the same way, if $y \in \overline{B}(x, r)$, then (1.7) is greater than or equal to 0, and

$$(1.9) \quad \overline{B}(y, t) \subseteq \overline{B}(x, r).$$

Let us say that $W \subseteq X$ is an *open set* with respect to $d(\cdot, \cdot)$ if for each $x \in W$ there is an $r > 0$ such that

$$(1.10) \quad B(x, r) \subseteq W.$$

The collection of open subsets of X with respect to $d(\cdot, \cdot)$ defines a topology on X , as usual. Every open ball in X with respect to $d(\cdot, \cdot)$ is an open set with respect to the topology determined on X by $d(\cdot, \cdot)$, by (1.8). If $d(\cdot, \cdot)$ is a metric on X , then X is Hausdorff with respect to the topology determined by $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is the discrete metric on X , then the corresponding topology on X is the same as the discrete topology, in which every subset of X is an open set.

Let $d(\cdot, \cdot)$ be any semimetric on X again, and put

$$(1.11) \quad V(x, r) = X \setminus \overline{B}(x, r) = \{y \in X : d(x, y) > r\}$$

for every $x \in X$ and $r \geq 0$. If $y \in V(x, r)$, then

$$(1.12) \quad t' = d(x, y) - r > 0,$$

and one can verify that

$$(1.13) \quad B(y, t') \subseteq V(x, r),$$

using the triangle inequality. This implies that $V(x, r)$ is an open set in X with respect to the topology determined by $d(\cdot, \cdot)$. Equivalently, this means that $\overline{B}(x, r)$ is a closed set in X with respect to this topology. Alternatively, one can use the triangle inequality to check that $\overline{B}(x, r)$ contains all of its limit points with respect to this topology.

2 Relations

Let X be a set again, and let $X \times X$ be the Cartesian product of X with itself, which is the set of ordered pairs (x, y) with $x, y \in X$. Thus a subset of $X \times X$ is the same as a (binary) *relation* on X . The *identity relation* on X corresponds to the *diagonal* in $X \times X$, defined by

$$(2.1) \quad \Delta = \Delta_X = \{(x, x) : x \in X\}.$$

If $U \subseteq X \times X$ is any relation on X , then put

$$(2.2) \quad \tilde{U} = \{(x, y) : (y, x) \in U\}.$$

This is sometimes denoted U^{-1} , and described as the inverse relation associated to U .

If $U, V \subseteq X \times X$ are relations on X , then let $U * V$ be the relation on X defined by

$$(2.3) \quad U * V = \{(x, z) \in X \times X : \text{there is a } y \in X \text{ such that} \\ (x, y) \in U \text{ and } (y, z) \in V\}.$$

This is the same as the composition $V \circ U$ of U and V in the notation and terminology on p7 of [12]. Note that

$$(2.4) \quad U * \Delta = \Delta * U = U$$

for every $U \subseteq X \times X$. If $U, V, W \subseteq X \times X$ are relations on X , then it is easy to see that

$$(2.5) \quad (U * V) * W = U * (V * W).$$

More precisely, both sides of (2.5) consist of the $(x, t) \in X \times X$ for which there are $y, z \in X$ such that $(x, y) \in U$, $(y, z) \in V$, and $(z, t) \in W$.

Suppose that $d(\cdot, \cdot)$ is a semimetric on X , and put

$$(2.6) \quad U(r) = U_d(r) = \{(x, y) \in X \times X : d(x, y) < r\}$$

for every $r > 0$. Observe that

$$(2.7) \quad \Delta \subseteq U(r)$$

for each $r > 0$, by (1.1), and that

$$(2.8) \quad \widetilde{U(r)} = U(r)$$

for every $r > 0$, by (1.2). The triangle inequality (1.3) implies that

$$(2.9) \quad U(r_1) * U(r_2) \subseteq U(r_1 + r_2)$$

for every $r_1, r_2 > 0$. By construction,

$$(2.10) \quad U(r) \subseteq U(t)$$

when $r < t$, and

$$(2.11) \quad \bigcap_{r>0} U(r) = \{(x, y) \in X \times X : d(x, y) = 0\}.$$

In particular,

$$(2.12) \quad \bigcap_{r>0} U(r) = \Delta$$

exactly when $d(\cdot, \cdot)$ is a metric on X .

If $U \subseteq X \times X$ is a relation on X and A is a subset of X , then we put

$$(2.13) \quad U[A] = \{y \in X : \text{there is an } x \in A \text{ such that } (x, y) \in U\}.$$

If $V \subseteq X \times X$ is another relation on X , then it is easy to see that

$$(2.14) \quad (U * V)[A] = V[U[A]].$$

In this case, $U \cap V$ is a relation on X too, and

$$(2.15) \quad (U \cap V)[A] \subseteq (U[A]) \cap (V[A]).$$

If $x \in X$, then it will be convenient to put

$$(2.16) \quad U[x] = U[\{x\}] = \{y \in X : (x, y) \in U\}.$$

Using this notation, we have equivalently that

$$(2.17) \quad U[A] = \bigcup_{x \in A} U[x]$$

for each $A \subseteq X$. Note that

$$(2.18) \quad (U \cap V)[x] = (U[x]) \cap (V[x]),$$

which is to say that equality holds in (2.15) when A has only one element. Let $d(\cdot, \cdot)$ be a semimetric on X again, and let $U(r)$ be as in (2.6) for some $r > 0$. Using the notation (2.16) with $U = U(r)$, we get that

$$(2.19) \quad (U(r))[x] = B(x, r)$$

for every $x \in X$, where $B(x, r)$ is as in (1.5).

3 Uniform structures

Let X be a set again, and let \mathcal{U} be a nonempty collection of subsets of $X \times X$. If \mathcal{U} satisfies the following five conditions, then \mathcal{U} is said to define a *uniformity* on X , and (X, \mathcal{U}) is said to be a *uniform space*, as on p176 of [12]. First, for each $U \in \mathcal{U}$, we should have that

$$(3.1) \quad \Delta \subseteq U,$$

where Δ is as in (2.1). Second, for each $U \in \mathcal{U}$, we ask that

$$(3.2) \quad \tilde{U} \in \mathcal{U}$$

too, where \tilde{U} is as in (2.2). Third, for every $U \in \mathcal{U}$, there should be a $V \in \mathcal{U}$ such that

$$(3.3) \quad V * V \subseteq U,$$

where $V * V$ is as defined in (2.3). Fourth, for any two elements U, V of \mathcal{U} , we ask that

$$(3.4) \quad U \cap V \in \mathcal{U}$$

as well. The fifth condition is that if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then we also have that

$$(3.5) \quad V \in \mathcal{U}.$$

This implies that $X \times X$ should be an element of \mathcal{U} , since \mathcal{U} is supposed to be nonempty.

Let $d(\cdot, \cdot)$ be a semimetric on X , and let $U_d(r) \subseteq X \times X$ be as in (2.6) for each $r > 0$. Also let \mathcal{U}_d be the collection of subsets U of $X \times X$ for which there is an $r > 0$ such that

$$(3.6) \quad U_d(r) \subseteq U.$$

It is easy to see that this defines a uniformity on X , using (2.7), (2.8), (2.9), and (2.10). More precisely, one can check that \mathcal{U}_d satisfies the first four requirements of a uniformity using these properties of $U_d(r)$. The fifth requirement of a uniformity is built into the definition of \mathcal{U}_d . If $d(\cdot, \cdot)$ is the discrete metric on X , then

$$(3.7) \quad U_d(r) = \Delta$$

when $0 < r \leq 1$. In this case, \mathcal{U}_d consists of all subsets of $X \times X$ that contain the diagonal Δ as a subset. If instead $d(x, y) = 0$ for every $x, y \in X$, then

$$(3.8) \quad U_d(r) = X \times X$$

for every $r > 0$, and $X \times X$ is the only element of \mathcal{U}_d .

Let \mathcal{U} be any uniformity on X . A subcollection \mathcal{B} of \mathcal{U} is said to be a *base* for \mathcal{U} if for each $U \in \mathcal{U}$ there is a $V \in \mathcal{B}$ such that

$$(3.9) \quad V \subseteq U.$$

In this case, \mathcal{U} is exactly the same as the collection of subsets U of $X \times X$ for which there is a $V \in \mathcal{B}$ that satisfies (3.9), because of the fifth requirement of a uniformity. In particular, \mathcal{U} is uniquely determined by \mathcal{B} under these conditions. If $d(\cdot, \cdot)$ is a semimetric on X , and if \mathcal{B}_d is the collection of subsets of $X \times X$ of the form $U_d(r)$ as in (2.6) for some $r > 0$, then \mathcal{B}_d is a base for the uniformity \mathcal{U}_d on X associated to d as in the preceding paragraph.

If \mathcal{B} is a base for a uniformity \mathcal{U} on X , then \mathcal{B} is a nonempty collection of subsets of $X \times X$, because $\mathcal{U} \neq \emptyset$. Each $U \in \mathcal{B}$ has to contain the diagonal Δ as a subset, because of (3.1). If $U \in \mathcal{B}$, then there should be a $V_0 \in \mathcal{B}$ such that

$$(3.10) \quad V_0 \subseteq \tilde{U},$$

by (3.2). There should also be a $V \in \mathcal{B}$ that satisfies (3.3). If $U, V \in \mathcal{B}$, then there should be a $W \in \mathcal{B}$ such that

$$(3.11) \quad W \subseteq U \cap V,$$

because of (3.4). Conversely, if \mathcal{B} is a nonempty collection of subsets of $X \times X$ that satisfies these four conditions, then one can check that

$$(3.12) \quad \mathcal{U}(\mathcal{B}) = \{U \subseteq X \times X : \text{there is a } V \in \mathcal{B} \text{ such that } V \subseteq U\}$$

is a uniformity on X . Note that \mathcal{B} is automatically a base for (3.12), by construction. If \mathcal{B} is a base for any uniformity \mathcal{U} on X , then \mathcal{U} has to be the same as (3.12), as in the previous paragraph.

4 The associated topology

Let (X, \mathcal{U}) be a uniform space. Let us say that $A \subseteq X$ is an *open set* with respect to the uniformity \mathcal{U} if for each $x \in A$ there is a $U \in \mathcal{U}$ such that

$$(4.1) \quad U[x] \subseteq A,$$

where $U[x]$ is as defined in (2.16). Note that the empty set and X itself are automatically open sets with respect to \mathcal{U} . The union of any collection of open subsets of X with respect to \mathcal{U} is clearly open with respect to \mathcal{U} too. Suppose that $A, B \subseteq X$ are open sets with respect to \mathcal{U} , and let us check that $A \cap B$ is open with respect to \mathcal{U} as well. Let $x \in A \cap B$ be given, and let U, V be elements of \mathcal{U} such that (4.1) holds and similarly

$$(4.2) \quad V[x] \subseteq B.$$

The fourth condition (3.4) in the definition of a uniformity implies that $U \cap V$ is an element of \mathcal{U} , and we have that

$$(4.3) \quad (U \cap V)[x] = (U[x]) \cap (V[x]) \subseteq A \cap B,$$

using (2.18) in the first step, and (4.1), (4.2) in the second step. This shows that $A \cap B$ is an open set in X with respect to \mathcal{U} , as desired. It follows that the collection of open subsets of X with respect to \mathcal{U} defines a topology on X .

Let $d(\cdot, \cdot)$ be a semimetric on a set X , and let \mathcal{U}_d be the corresponding uniformity on X , as in the previous section. In this case, the open subsets of X with respect to \mathcal{U}_d are the same as the open subsets of X with respect to the topology determined by $d(\cdot, \cdot)$ in the usual way, as in Section 1. This can be

verified using (2.19) and the relevant definitions. In particular, if $d(\cdot, \cdot)$ is the discrete metric on X , then \mathcal{U}_d consists of all subsets of $X \times X$ that contain the diagonal Δ , and the corresponding topology on X is the discrete topology. If instead $d(x, y) = 0$ for every $x, y \in X$, then $X \times X$ is the only element of \mathcal{U}_d , and the corresponding topology on X is the discrete topology.

Let (X, \mathcal{U}) be any uniform space again, and let A be a subset of X . Put

$$(4.4) \quad A_0 = \{x \in X : \text{there is a } U \in \mathcal{U} \text{ such that } U[x] \subseteq A\}.$$

Note that

$$(4.5) \quad x \in U[x]$$

for every $x \in X$ and $U \in \mathcal{U}$, because of (3.1). Thus

$$(4.6) \quad A_0 \subseteq A$$

automatically. As in Theorem 4 on p178 of [12],

$$(4.7) \quad A_0 = \text{Int}(A),$$

where $\text{Int}(A)$ denotes the interior of A with respect to the topology on X associated to \mathcal{U} . More precisely, it is easy to see that

$$(4.8) \quad \text{Int}(A) \subseteq A_0,$$

directly from the definitions. In order to show that (4.7) holds, it suffices to verify that A_0 is an open set in X with respect to the topology associated to \mathcal{U} . Let $x \in A_0$ be given, and let U be an element of \mathcal{U} such that (4.1) holds, as in (4.4). By definition of a uniformity, there is a $V \in \mathcal{U}$ that satisfies (3.3). It follows that

$$(4.9) \quad (V * V)[x] \subseteq U[x] \subseteq A.$$

This implies that

$$(4.10) \quad V[V[x]] = (V * V)[x] \subseteq A,$$

using (2.14) in the first step. Equivalently, this means that

$$(4.11) \quad V[y] \subseteq A$$

for every $y \in V[x]$, as in (2.17). Thus $y \in A_0$ for each $y \in V[x]$, so that

$$(4.12) \quad V[x] \subseteq A_0.$$

This shows that A_0 is an open set in X with respect to the topology associated to \mathcal{U} , as desired, because $x \in A_0$ is arbitrary.

In particular, if x is any element of X and U is any element of \mathcal{U} , then

$$(4.13) \quad x \in \text{Int}(U[x]).$$

This follows by applying (4.7) to $A = U[x]$, in which case $x \in A_0$ automatically. Now let $A \subseteq X$ and $U \in \mathcal{U}$ be given, and put

$$(4.14) \quad B = \bigcup_{x \in A} \text{Int}(U[x]).$$

Thus B is automatically an open set in X with respect to the topology associated to \mathcal{U} , since it is a union of open sets. We also have that

$$(4.15) \quad A \subseteq B \subseteq U[A],$$

where $U[A]$ is defined in (2.13). More precisely, the first inclusion in (4.15) uses (4.13). The second inclusion in (4.15) uses the equivalent expression for $U[A]$ in (2.17), and the fact that the interior of $U[x]$ is automatically contained in $U[x]$.

5 Closure and regularity

Let (X, \mathcal{U}) be a uniform space, let A be a subset of X , and let \overline{A} be the closure of A in X with respect to the topology associated to \mathcal{U} . Let us check that

$$(5.1) \quad \overline{A} = \{x \in X : U[x] \cap A \neq \emptyset \text{ for every } U \in \mathcal{U}\},$$

where $U[x]$ is as in (2.16). Of course, \overline{A} can normally be defined as the set of $x \in X$ such that every open subset of X that contains x also intersects A . The fact that the right side of (5.1) is contained in \overline{A} thus follows directly from the definition of the topology on X associated to \mathcal{U} . The opposite inclusion can be obtained similarly, using (4.13).

Now let us use (5.1) to verify that

$$(5.2) \quad \overline{A} = \bigcap_{V \in \mathcal{U}} V[A],$$

where $V[A]$ is as in (2.13). This is the same as saying that

$$(5.3) \quad \overline{A} = \bigcap_{U \in \mathcal{U}} \widetilde{U}[A],$$

where \widetilde{U} is as defined in (2.2). The equivalence of (5.2) and (5.3) uses the second condition (3.2) in the definition of a uniformity, and the simple fact that

$$(5.4) \quad \widetilde{(\widetilde{U})} = U$$

for every $U \subseteq X \times X$. It is easy to see that

$$(5.5) \quad \widetilde{U}[A] = \{x \in X : U[x] \cap A \neq \emptyset\}$$

for every $U \subseteq X \times X$, directly from the definitions. Thus (5.3) is a reformulation of (5.1), as desired.

Remember that a topological space is said to be *regular* in the strict sense if for each point x and closed set E in the space with $x \notin E$ there are disjoint open sets that contain x and E . This is equivalent to saying that for each point x in the space and open set W that contains x there is an open set W_1 that contains x and whose closure is contained in W . If the space also satisfies the first or even 0th separation condition, then it follows that the space is Hausdorff. Sometimes one may include the first or 0th separation condition in the definition of regularity of a topological space, and so we refer to regularity in the strict sense to indicate that the first or 0th separation condition is not necessarily included. One may also say that a topological space satisfies the *third separation condition* when it is regular in the strict sense and satisfies the first or 0th separation condition, and hence is Hausdorff.

Let (X, \mathcal{U}) be a uniform space again, and let us check that X is regular in the strict sense with respect to the topology associated to \mathcal{U} . Let $x \in X$ be given, and suppose that $A \subseteq X$ is an open set with $x \in A$. Thus there is a $U \in \mathcal{U}$ such that

$$(5.6) \quad U[x] \subseteq A,$$

as in the previous section. The third condition in the definition of a uniformity implies that there is a $V \in \mathcal{U}$ that satisfies (3.3), so that

$$(5.7) \quad V[V[x]] = (V * V)[x] \subseteq U[x] \subseteq A.$$

It follows that

$$(5.8) \quad \overline{V[x]} \subseteq V[V[x]] \subseteq A,$$

using (5.2) applied to $V[x]$ in the first step. In particular,

$$(5.9) \quad \overline{\text{Int}(V[x])} \subseteq A,$$

where $\text{Int}(V[x])$ is the interior of $V[x]$ in X , as before. Of course, $\text{Int}(V[x])$ is an open set in X that contains x , by (4.13). Hence (5.9) implies that X is regular in the strict sense, as desired.

Here is a slightly different version of the same type of argument. Let $x \in X$ and a closed set $E \subseteq X$ be given, with $x \notin E$. Thus $A = X \setminus E$ is an open set that contains x , and so there is a $U \in \mathcal{U}$ that satisfies (5.6) as before. This leads to a $V \in \mathcal{U}$ that satisfies (5.7), which can be reformulated as saying that

$$(5.10) \quad V[x] \cap (\tilde{V}[E]) = \emptyset.$$

Remember that $\tilde{V} \in \mathcal{U}$ too, by the second condition (3.2) in the definition of a uniformity. It follows that there is an open subset of X that contains E and is contained in $\tilde{V}[E]$, as in (4.14) and (4.15). We have also seen that $x \in \text{Int}(V[x])$, so that (5.10) implies that x and E are contained in disjoint open subsets of X .

Now let $x, y \in X$ be given, with $x \neq y$, and suppose that there is a $U \in \mathcal{U}$ such that

$$(5.11) \quad y \notin U[x].$$

Let V be an element of \mathcal{U} that satisfies (3.3), so that

$$(5.12) \quad V[V[x]] = (V * V)[x] \subseteq U[x],$$

using (2.14) in the first step. Thus (5.11) implies that $y \notin V[V[x]]$, which is the same as saying that

$$(5.13) \quad V[x] \cap (\tilde{V}[y]) = \emptyset,$$

where \tilde{V} is as in (2.2). In particular, (5.13) implies that the interiors of $V[x]$ and $\tilde{V}[x]$ are disjoint. Of course, we already know that x and y are contained in the interiors of $V[x]$ and $\tilde{V}[y]$, respectively, as in (4.13). This is basically a more direct version of the proof that X is Hausdorff when X satisfies the first or 0th separation condition, since we already know that X is regular in the strict sense. This can also be formulated as saying that X is Hausdorff when

$$(5.14) \quad \bigcap_{U \in \mathcal{U}} U = \Delta,$$

where Δ is the diagonal (2.1) in $X \times X$. Conversely, if X satisfies the first separation condition, then it is easy to see that (5.14) holds. This also works with the 0th separation condition, using the second condition (3.2) in the definition of a uniformity.

This discussion applies in particular to the case where the topology on X is determined by a semimetric $d(\cdot, \cdot)$, as in Section 1. In this case, regularity in the strict sense can be derived from the fact that closed balls are closed sets, or equivalently that subsets of X as in (1.11) are open. Of course, one can check directly that X is Hausdorff when $d(\cdot, \cdot)$ is a metric on X .

6 Symmetry and interior

Let X be a set. As usual, a set $U \subseteq X \times X$ is said to be *symmetric* if

$$(6.1) \quad \tilde{U} = U,$$

where \tilde{U} is as in (2.2). If U is any subset of $X \times X$, then

$$(6.2) \quad V = U \cap \tilde{U}$$

is automatically symmetric. If \mathcal{U} is a uniformity on X and $U \in \mathcal{U}$, then (6.2) is an element of \mathcal{U} as well. It follows that the symmetric elements of \mathcal{U} form a base for \mathcal{U} .

If U is any subset of $X \times X$ and $x \in X$, then $U[x]$ was defined in (2.16). Thus, for each $y \in X$, we have that

$$(6.3) \quad \tilde{U}[y] = \{x \in X : (y, x) \in \tilde{U}\} = \{x \in X : (x, y) \in U\}.$$

If V is another subset of $X \times X$, then $U * V$ is defined in (2.3), and can be given equivalently as

$$(6.4) \quad U * V = \bigcup_{y \in X} \tilde{U}[y] \times V[y].$$

If $W \subseteq X \times X$ too, then

$$(6.5) \quad (U * V) * W = U * (V * W) = \bigcup_{(y,z) \in V} \tilde{U}[y] \times W[z],$$

as in (2.5). This is a reformulation of Lemma 1 on p176 of [12], and one may drop the parentheses on the left side, because of associativity.

Let \mathcal{U} be a uniformity on X again, let $U \in \mathcal{U}$ be given, and let V be an element of \mathcal{U} that satisfies (3.3). This implies that

$$(6.6) \quad \bigcup_{y \in X} \tilde{V}[y] \times V[y] \subseteq U,$$

as in (6.4). Remember that y is an element of the interior of $V[y]$ with respect to the topology on X associated to \mathcal{U} for each $y \in X$, as in (4.13). Similarly, y is an element of the interior of $\tilde{V}[y]$ for every $y \in X$, because $\tilde{V} \in \mathcal{U}$ too, as in (3.2). It follows that (y, y) is an element of the interior of $\tilde{V}[y] \times V[y]$ with respect to the corresponding product topology on $X \times X$ for every $y \in X$. This shows that the diagonal Δ is contained in the interior of U with respect to the product topology on $X \times X$. As a refinement of this argument, one can first find a $W \in \mathcal{U}$ such that

$$(6.7) \quad W * W * W \subseteq U.$$

Using (6.5), we get that

$$(6.8) \quad \bigcup_{(y,z) \in W} \tilde{W}[y] \times W[z] \subseteq U,$$

so that W is contained in the interior of U with respect to the product topology on $X \times X$.

Suppose that $E \subseteq X \times X$, and let \overline{E} be the closure of E with respect to the product topology on $X \times X$, using the topology on X associated to \mathcal{U} . One can check that

$$(6.9) \quad \overline{E} = \bigcap_{V_1, V_2 \in \mathcal{U}} V_1 * E * V_2,$$

in analogy with (5.2), and using (6.5). In particular, if $W \in \mathcal{U}$ is as in (6.7), then we get that

$$(6.10) \quad \overline{W} \subseteq U.$$

7 Compactness

Let (X, \mathcal{U}) be a uniform space, and suppose that $K \subseteq X$ is compact with respect to the topology on X associated to \mathcal{U} as in Section 4. Let $W \subseteq X$ be an open set with respect to the associated topology that contains K as a subset. We would like to check that there is a $V \in \mathcal{U}$ such that

$$(7.1) \quad V[K] \subseteq W,$$

where $V[K]$ is as defined in (2.13). Because $K \subseteq W$ and W is an open set in X , we have that for each $x \in K$ there is a $U_x \in \mathcal{U}$ such that

$$(7.2) \quad U_x[x] \subseteq W,$$

as in (4.1). Using the definition of a uniform space, we get that for each $x \in K$ there is a $V_x \in \mathcal{U}$ that satisfies

$$(7.3) \quad V_x * V_x \subseteq U_x,$$

as in (3.3). Thus

$$(7.4) \quad V_x[V_x[x]] = (V_x * V_x)[x] \subseteq U_x[x] \subseteq W$$

for every $x \in K$, using (2.14) in the first step. Remember that x is an element of the interior of $V_x[x]$ in X for each $x \in K$, as in (4.13). It follows that there are finitely many elements x_1, \dots, x_n of K such that

$$(7.5) \quad K \subseteq \bigcup_{j=1}^n V_{x_j}[x_j],$$

because K is compact in X . Put

$$(7.6) \quad V = \bigcap_{j=1}^n V_{x_j},$$

which is an element of \mathcal{U} , since $V_{x_j} \in \mathcal{U}$ for each $j = 1, \dots, n$. Observe that

$$(7.7) \quad V[V_{x_j}[x_j]] \subseteq V_{x_j}[V_{x_j}[x_j]] \subseteq W$$

for each $j = 1, \dots, n$, using the fact that $V \subseteq V_{x_j}$ for each j in the first step, and (7.4) in the second step. Combining this with (7.5), we get that

$$(7.8) \quad V[K] \subseteq \bigcup_{j=1}^n V[V_{x_j}[x_j]] \subseteq W,$$

as desired.

If V is any element of \mathcal{U} , then there is a $V_1 \in \mathcal{U}$ such that

$$(7.9) \quad V_1 * V_1 \subseteq V,$$

as in (3.3). This implies that

$$(7.10) \quad V_1[V_1[K]] = (V_1 * V_1)[K] \subseteq V[K]$$

for any $K \subseteq X$, using (2.14) in the first step. It follows that

$$(7.11) \quad \overline{V_1[K]} \subseteq V_1[V_1[K]] \subseteq V[K],$$

using (5.2) in the first step. If V , K , and W are as in (7.1), then we get that

$$(7.12) \quad \overline{V_1[K]} \subseteq W.$$

Remember that K is automatically contained in the interior of $V_1[K]$ for every $V_1 \in \mathcal{U}$, as in (4.15).

Suppose now that X is a topological space which is regular in the strict sense, as in Section 5. If $K \subseteq X$ is compact, $W \subseteq X$ is an open set, and $K \subseteq W$, then there is an open subset of X that contains K and whose closure is contained in W . More precisely, one can first use regularity to cover K by open sets whose closures are contained in W , and then use compactness to reduce to a finite subcover.

8 Semi-ultrametrics

A semimetric $d(\cdot, \cdot)$ on a set X is said to be a *semi-ultrametric* on X if it satisfies

$$(8.1) \quad d(x, z) \leq \max(d(x, y), d(y, z))$$

for every $x, y, z \in X$. Of course, (8.1) automatically implies the ordinary version (1.3) of the triangle inequality. If a semi-ultrametric $d(\cdot, \cdot)$ on X is also a metric on X , so that $d(\cdot, \cdot)$ satisfies (1.4) too, then $d(\cdot, \cdot)$ is said to be an *ultrametric* on X . It is easy to see that the discrete metric on any set is an ultrametric.

Suppose that $d(\cdot, \cdot)$ is a semi-ultrametric on a set X , and let $r > 0$ be given. Observe that

$$(8.2) \quad d(x, y) < r$$

defines an equivalence relation on X . The corresponding equivalence class in X containing a point $x \in X$ is the same as the open ball $B(x, r)$ centered at x with radius r with respect to $d(\cdot, \cdot)$, as in (1.5). If $x, y \in X$ satisfy (8.2), then the equivalence classes containing x and y are the same, so that

$$(8.3) \quad B(x, r) = B(y, r).$$

Similarly, any two open balls in X of radius r with respect to $d(\cdot, \cdot)$ are either the same or disjoint as subsets of X , because X is partitioned by these equivalence classes. In particular, the complement of any open ball in X of radius r with respect to $d(\cdot, \cdot)$ can be expressed as a union of other open balls of radius r . This implies that for every $x \in X$,

$$(8.4) \quad B(x, r) \text{ is a closed set in } X$$

with respect to the usual topology on X associated to $d(\cdot, \cdot)$, because its complement is an open set.

In the same way,

$$(8.5) \quad d(x, y) \leq r$$

defines an equivalence relation on X for each $r \geq 0$. As before, the corresponding equivalence class in X containing a point $x \in X$ is the same as the closed ball

$\overline{B}(x, r)$ centered at x with radius r with respect to $d(\cdot, \cdot)$, which was defined in (1.6). If $x, y \in X$ satisfy (8.5), then the equivalence classes containing x and y are the same, which means that

$$(8.6) \quad \overline{B}(x, r) = \overline{B}(y, r).$$

It follows that

$$(8.7) \quad \overline{B}(x, r) \text{ is an open set in } X$$

for every $x \in X$ when $r > 0$, with respect to the usual topology on X associated to $d(\cdot, \cdot)$. Note that if $r = 0$, then (8.5) defines an equivalence relation on X for every semi-metric $d(\cdot, \cdot)$ on X .

Now let $x \sim y$ be any equivalence relation on a set X , and define $d(x, y)$ for $x, y \in X$ by

$$(8.8) \quad \begin{aligned} d(x, y) &= 0 && \text{when } x \sim y \\ &= 1 && \text{when } x \not\sim y. \end{aligned}$$

One can check that this defines a semi-ultrametric on X , which we shall call the *discrete semi-ultrametric* associated to this equivalence relation. This reduces to the discrete metric on X when this equivalence relation is simply equality of elements of X . If $d(x, y)$ is any semimetric on X that only takes the values 0 or 1, then it is easy to see that $d(x, y)$ is a semi-ultrametric on X . In this case, $d(x, y)$ is the same as the discrete semi-ultrametric associated to the equivalence relation (8.2) for any $0 < r \leq 1$, which is the same as the equivalence relation (8.5) for any $0 \leq r < 1$.

9 Uniform continuity

Let (X, \mathcal{U}) and (Y, \mathcal{V}) be uniform spaces, and let f be a mapping from X into Y . This leads to a mapping $f_2 : X \times X \rightarrow Y \times Y$ defined by

$$(9.1) \quad f_2(x, x') = (f(x), f(x'))$$

for every $x, x' \in X$. We say that f is *uniformly continuous* if for each $V \in \mathcal{V}$ we have that

$$(9.2) \quad f_2^{-1}(V) \in \mathcal{U}.$$

Equivalently, this means that for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that

$$(9.3) \quad f_2(U) \subseteq V.$$

If \mathcal{U} and \mathcal{V} are determined by semimetrics on X and Y , respectively, as in Section 3, then uniform continuity can also be characterized equivalently in terms of ϵ 's and δ 's in the usual way. In the case of arbitrary uniformities, it suffices to verify (9.2) for all V in a base for \mathcal{V} , in order to show that f is uniformly continuous. There is an analogous statement for sub-bases for \mathcal{V} , which will be defined in Section 14.

Let $U \subseteq X \times X$ be given, and put

$$(9.4) \quad U[x] = \{x' \in X : (x, x') \in U\}$$

for each $x \in X$, as in (2.16). Thus

$$(9.5) \quad U = \bigcup_{x \in X} (\{x\} \times U[x]),$$

which implies that

$$(9.6) \quad f_2(U) = \bigcup_{x \in X} (\{f(x)\} \times f(U[x])).$$

If $V \subseteq Y \times Y$, then it follows that (9.3) holds if and only if

$$(9.7) \quad f(U[x]) \subseteq V[f(x)]$$

for every $x \in X$.

Let X and Y be equipped with the topologies associated to the uniformities \mathcal{U} and \mathcal{V} , respectively, as in Section 4. One can check that a mapping $f : X \rightarrow Y$ is continuous at a point $x \in X$ with respect to these topologies on X and Y if and only if for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that (9.7) holds. This uses (4.13) and its analogue in Y , in addition to the basic definitions. If f is uniformly continuous, then it follows that f is continuous in the ordinary sense, because of the relationship between (9.3) and (9.7).

As a slight variant of this argument, remember that $f : X \rightarrow Y$ is continuous if and only if for every open subset W of Y , $f^{-1}(W)$ is an open subset of X . In this situation, $f^{-1}(W)$ is an open subset of X if and only if for each $x \in f^{-1}(W)$ there is a $U \in \mathcal{U}$ such that

$$(9.8) \quad U[x] \subseteq f^{-1}(W).$$

Of course, if $x \in f^{-1}(W)$, then $f(x) \in W$. Because W is supposed to be an open set in Y , there should be a $V \in \mathcal{V}$ such that

$$(9.9) \quad V[f(x)] \subseteq W.$$

If there is a $U \in \mathcal{U}$ that satisfies (9.7), then we get that

$$(9.10) \quad f(U[x]) \subseteq W,$$

which is equivalent to (9.8).

As another variant, let $x \in X$ and $V \subseteq Y \times Y$ be given, and observe that

$$(9.11) \quad \begin{aligned} f^{-1}(V[f(x)]) &= \{x' \in X : f(x') \in V[f(x)]\} \\ &= \{x' \in X : (f(x), f(x')) \in V\} = (f_2^{-1}(V))[x]. \end{aligned}$$

This permits one to deal with uniform continuity in terms of (9.2), as in the proof of Theorem 9 on p181 of [12].

Let (Z, \mathcal{W}) be another uniform space, and let g be a mapping from Y into Z . Also let g_2 be the corresponding mapping from $Y \times Y$ into $Z \times Z$, as in

(9.1). The composition $g \circ f$ of f and g maps X into Z , and leads to a mapping $(g \circ f)_2$ from $X \times X$ into $Z \times Z$ as in (9.1) as well. It is easy to see that

$$(9.12) \quad (g \circ f)_2 = g_2 \circ f_2$$

as mappings from $X \times X$ into $Z \times Z$. If f and g are both uniformly continuous, then one can use (9.12) that $g \circ f$ is uniformly continuous from X into Z .

Let us say that $f : X \rightarrow Y$ is uniformly continuous *along* a subset E of X if for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that (9.7) holds for every $x \in E$. Equivalently, this means that

$$(9.13) \quad (f(x), f(x')) \in V$$

for every $(x, x') \in U$ such that $x \in E$. This holds with $E = X$ if and only if f is uniformly continuous as a mapping from X into Y , as before. If f is uniformly continuous along any set $E \subseteq X$, then f is continuous as a mapping from X into Y at every point in E , with respect to the topologies associated to the given uniformities. This follows from the earlier discussion of the continuity of f at a point $x \in X$ in terms of (9.7). This condition also implies that the restriction of f to E is uniformly continuous with respect to the uniformity induced on E by \mathcal{U} on X , which is defined in Section 12. If f is uniformly continuous along $E \subseteq X$ and $g : Y \rightarrow Z$ is uniformly continuous along $f(E) \subseteq Y$, then it is easy to see that $g \circ f$ is uniformly continuous along E too, as a mapping from X into Z .

10 Compactness, continued

Let (X, \mathcal{U}) and (Y, \mathcal{V}) be uniform spaces again, and suppose that f is a continuous mapping from X into Y with respect to their associated topologies. If $K \subseteq X$ is compact, then we would like to verify that f is uniformly continuous along K , as in the preceding section. In particular, if X is compact, then it follows that f is uniformly continuous as a mapping from X into Y .

Let $V \in \mathcal{V}$ be given, and let V_1 be an element of \mathcal{V} such that

$$(10.1) \quad \widetilde{V}_1 * V_1 \subseteq V.$$

More precisely, one might as well choose $V_1 \in \mathcal{V}$ to be symmetric and to satisfy $V_1 * V_1 \subseteq V$, using (3.2), (3.3), and (3.4). Also let $x \in K$ be given, and remember that $f(x)$ is an element of the interior of $V_1[f(x)]$ in Y , as in (4.13). Because f is continuous at x , there is a $U_{1,x} \in \mathcal{U}$ such that

$$(10.2) \quad f(U_{1,x}[x]) \subseteq V_1[f(x)],$$

as in (9.7) and discussed in the paragraph immediately after that. Let $U_{2,x}$ be an element of \mathcal{U} such that

$$(10.3) \quad U_{2,x} * U_{2,x} \subseteq U_{1,x},$$

as in (3.3) again. If K is compact in X , then there are finitely many elements x_1, \dots, x_n of K such that

$$(10.4) \quad K \subseteq \bigcup_{j=1}^n U_{2,x_j}[x_j].$$

This uses the fact that x is an element of the interior of $U_{2,x}[x]$ in X for every $x \in K$, as in (4.13). Put

$$(10.5) \quad U_2 = \bigcup_{j=1}^n U_{2,x_j},$$

which is an element of \mathcal{U} , since $U_{2,x_j} \in \mathcal{U}$ for every $j = 1, \dots, n$. We would like to check that

$$(10.6) \quad (f(x), f(x')) \in V$$

for every $x \in K$ and $x' \in X$ such that $(x, x') \in U_2$, as in (9.13). Let $x \in K$ be given again, and choose $j \in \{1, \dots, n\}$ such that

$$(10.7) \quad x \in U_{2,x_j}[x_j],$$

which is possible by (10.4). If $x' \in X$ satisfies $(x, x') \in U_2 \subseteq U_{2,x_j}$, then

$$(10.8) \quad x' \in U_{2,x_j}[U_{2,x_j}[x_j]] = (U_{2,x_j} * U_{2,x_j})[x_j] \subseteq U_{1,x_j}[x_j],$$

using (2.14) in the second step, and (10.3) in the third step. Note that

$$(10.9) \quad U_{2,x_j} \subseteq U_{1,x_j},$$

because of (10.3) and the fact that $\Delta \subseteq U_{2,x_j}$, as in (3.1). Thus (10.7) implies that

$$(10.10) \quad x \in U_{1,x_j}[x_j].$$

Using (10.8) and (10.10), we can apply (10.2) to x_j to get that

$$(10.11) \quad f(x), f(x') \in V_1[f(x_j)].$$

This implies that

$$(10.12) \quad (f(x), f(x')) \in \widetilde{V}_1 * V_1 \subseteq V,$$

using (10.1) in the second step. Thus (10.6) holds, as desired.

11 Totally bounded sets

Let (X, \mathcal{U}) be a uniform space. A subset E of X is said to be *totally bounded* if for each $U \in \mathcal{U}$ there is a finite set $A \subseteq X$ such that

$$(11.1) \quad E \subseteq U[A],$$

where $U[A]$ is as in (2.13). Equivalently, this means that there are finitely many elements x_1, \dots, x_n of X such that

$$(11.2) \quad E \subseteq \bigcup_{j=1}^n U[x_j],$$

where $U[x_j]$ is as in (2.16). If E is compact with respect to the topology on X associated to \mathcal{U} as in Section 4, then E is totally bounded. Indeed, if $U \in \mathcal{U}$, then we can cover E by open subsets of X of the form $\text{Int}(U[x])$ with $x \in E$, because of (4.13), and then reduce to a finite subcovering to get (11.2).

Let $V \subseteq X \times X$ be given, and let us say that $B \subseteq X$ is *V-small* if

$$(11.3) \quad (x, y) \in V$$

for every $x, y \in B$, which implies that every subset of B is *V-small* too. Equivalently, B is *V-small* when

$$(11.4) \quad B \times B \subseteq V.$$

which is the same as saying that

$$(11.5) \quad B \subseteq V[x]$$

for every $x \in B$. Suppose that $U \subseteq X \times X$ satisfies

$$(11.6) \quad \tilde{U} * U \subseteq V,$$

using the notation in (2.2) and (2.3). If

$$(11.7) \quad B \subseteq U[w]$$

for some $w \in X$, then it is easy to see that B is *V-small* in X . Using this, one can check that $E \subseteq X$ is totally bounded if and only if for each $V \in \mathcal{U}$ there are finitely many *V-small* subsets of X whose union contains E . More precisely, the “if” part of this statement follows from (11.5). In the other direction, if $V \in \mathcal{U}$ is given, then there is a $U \in \mathcal{U}$ that satisfies (11.6), by the definition of a uniformity. With this choice of U , (11.2) implies that E can be covered by finitely many *V-small* subsets of X .

Note that V , \tilde{V} , and $V \cap \tilde{V}$ determine the same collection of small subsets of X for any $V \subseteq X \times X$. If $V_1, V_2 \subseteq X \times X$, then $B \subseteq X$ is small with respect to $V_1 \cap V_2$ if and only if B is small with respect to both V_1 and V_2 . In particular, if $B_1 \subseteq X$ is V_1 -small, and $B_2 \subseteq X$ is V_2 -small, then $B_1 \cap B_2$ is small with respect to $V_1 \cap V_2$. If $E \subseteq X$ can be covered by finitely many sets that are V_1 small, and by finitely many sets that are V_2 small, then it follows that E can be covered by finitely many sets that are small with respect to $V_1 \cap V_2$, by taking intersections of the various sets that are small with respect to V_1 and V_2 .

Let (Y, \mathcal{V}) be another uniform space, and suppose that $f : X \rightarrow Y$ is uniformly continuous. If $E \subseteq X$ is totally bounded, then one can check that $f(E)$ is totally bounded in Y .

12 Induced uniform structures

Let (Y, \mathcal{V}) be a uniform space, and let X be a subset of Y . Consider

$$(12.1) \quad \mathcal{U} = \{U \subseteq X \times X : \text{there is a } V \in \mathcal{V} \text{ such that } U = V \cap (X \times X)\}.$$

One can check that this is a uniformity on X , which may be described as the uniformity induced on X by \mathcal{V} on Y . By construction, the natural inclusion mapping from X into Y is uniformly continuous with respect to \mathcal{U} and \mathcal{V} . If $\mathcal{B}(\mathcal{V})$ is a base for \mathcal{V} , then it is easy to see that

$$(12.2) \quad \mathcal{B}(\mathcal{U}) = \{V \cap (X \times X) : V \in \mathcal{B}(\mathcal{V})\}$$

is a base for \mathcal{U} .

Note that the restriction of a semimetric $d(\cdot, \cdot)$ on Y to elements of X defines a semimetric on X . If \mathcal{V} is the uniformity determined on Y by $d(\cdot, \cdot)$ as in Section 3, then the induced uniformity on $X \subseteq Y$ is the same as the uniformity determined on X by the restriction of $d(\cdot, \cdot)$ to elements of X . One way to look at this is to use bases for these uniformities consisting of sets of the form (2.6) associated to $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a semi-ultrametric on Y , then the restriction of $d(\cdot, \cdot)$ to elements of X defines a semi-ultrametric on X too.

Let \mathcal{V} be any uniformity on Y again, and let \mathcal{U} be the induced uniformity on $X \subseteq Y$. This leads to associated topologies on X and Y , as in Section 4. The associated topology on Y also leads to an induced topology on X in the usual way, where $A \subseteq X$ is an open set if there is an open set $B \subseteq Y$ such that

$$(12.3) \quad A = B \cap X.$$

In this case, it is easy to see that A is an open set with respect to the topology on X associated to \mathcal{U} , just by unwinding the definitions. The converse is a bit more complicated, as in the context of semimetric spaces.

Suppose that $A \subseteq X$ is an open set with respect to the topology associated to \mathcal{U} . Thus for each $x \in A$ there is a $U_x \in \mathcal{U}$ such that

$$(12.4) \quad U_x[x] \subseteq A,$$

as in (4.1), and where $U_x[x]$ is as in (2.16). By definition of \mathcal{U} , for each $x \in A$ there is a $V_x \in \mathcal{V}$ such that

$$(12.5) \quad U_x = V_x \cap (X \times X).$$

This implies that

$$(12.6) \quad U_x[x] = V_x[x] \cap X$$

for each $x \in A$, where $V_x[x]$ is also defined as in (2.16), but in Y instead of X . Let $\text{Int}(V_x[x])$ be the interior of $V_x[x]$ in Y with respect to the topology associated to \mathcal{V} for each $x \in A$. Put

$$(12.7) \quad B = \bigcup_{x \in A} \text{Int}(V_x[x]),$$

which is also an open set in Y with respect to the topology associated to \mathcal{V} . Observe that $A \subseteq B$, because $x \in \text{Int}(V_x[x])$ for each $x \in A$, as in (4.13). It follows that $A \subseteq B \cap X$, since $A \subseteq X$ by hypothesis. To get the opposite inclusion, one can use (12.4), (12.6), and the fact that $V_x[x] \subseteq \text{Int}(V_x[x])$ for every $x \in A$ automatically.

13 Induced uniform structures, continued

Let X and Y be sets, and let f be a mapping from X into Y . This leads to a mapping f_2 from $X \times X$ into $Y \times Y$, as in (9.1). If $U, U' \subseteq X \times X$, then

$$(13.1) \quad f_2(\widetilde{U}) = \widetilde{f_2(U)}$$

and

$$(13.2) \quad f_2(U * U') \subseteq f_2(U) * f_2(U'),$$

where \widetilde{U} and $U * U'$ are as in (2.2) and (2.3). If f is injective, then equality holds in (13.2). Similarly, if $V, V' \subseteq Y \times Y$, then

$$(13.3) \quad f_2^{-1}(\widetilde{V}) = \widetilde{f_2^{-1}(V)}$$

and

$$(13.4) \quad f_2^{-1}(V) * f_2^{-1}(V') \subseteq f_2^{-1}(V * V').$$

If f is surjective, then equality holds in (13.4). Of course,

$$(13.5) \quad f_2(\Delta_X) \subseteq \Delta_Y,$$

where Δ_X, Δ_Y are as in (2.1).

If \mathcal{V} is a uniformity on Y , then one can check that

$$(13.6) \quad \mathcal{B} = \{f_2^{-1}(V) : V \in \mathcal{V}\}$$

is a base for a uniformity \mathcal{U} on X . Equivalently, \mathcal{U} consists of the $U \subseteq X \times X$ for which there is a $V \in \mathcal{V}$ such that

$$(13.7) \quad f_2^{-1}(V) \subseteq U.$$

By construction, f is uniformly continuous as a mapping from X into Y with respect to \mathcal{U} and \mathcal{V} , respectively. If f is injective, then (13.6) is already a uniformity on X . In particular, if $X \subseteq Y$ and f is the natural inclusion mapping from X into Y , then (13.6) is the same as the uniformity induced on X by \mathcal{V} on Y as in (12.1).

Let f be any mapping from a set X into a set Y again, and let \mathcal{B}_Y be a base for a uniformity \mathcal{V} on Y . In this case,

$$(13.8) \quad \mathcal{B}_X = \{f_2^{-1}(V) : V \in \mathcal{B}_Y\}$$

is a base for the same uniformity \mathcal{U} on X as in the previous paragraph. There is an analogous statement for sub-bases, which are defined in the next section. Note that a mapping from another uniform space into X is uniformly continuous with respect to \mathcal{U} on X if and only if the composition of this mapping with f is uniformly continuous as a mapping into Y , with respect to the given uniformity \mathcal{V} on Y . The “only if” part of this statement follows from the fact that compositions of uniformly continuous mappings are uniformly continuous, as in Section 9, while the “if” part uses the way that \mathcal{U} is defined in terms of \mathcal{V} here.

If $d_Y(\cdot, \cdot)$ is a semimetric on Y , then it is easy to see that

$$(13.9) \quad d_X(x, x') = d_Y(f(x), f(x'))$$

defines a semimetric on X . Put

$$(13.10) \quad U_{d_Y}(r) = \{(y, y') \in Y \times Y : d_Y(y, y') < r\}$$

and

$$(13.11) \quad U_{d_X}(r) = \{(x, x') \in X \times X : d_X(x, x') < r\}$$

for each $r > 0$, as in (2.6). If $f_2 : X \times Y \rightarrow Y \times Y$ is as in (9.1) again, then

$$(13.12) \quad f_2^{-1}(U_{d_Y}(r)) = U_{d_X}(r)$$

for every $r > 0$. Let \mathcal{B}_{d_Y} be the collection of subsets of $Y \times Y$ of the form (13.10) for some $r > 0$, and let \mathcal{B}_{d_X} be the collection of subsets of $X \times X$ of the form (13.11) for some $r > 0$, as in Section 3. These are bases for the uniformities \mathcal{U}_{d_Y} and \mathcal{U}_{d_X} on Y and X associated to $d_Y(\cdot, \cdot)$ and $d_X(\cdot, \cdot)$, respectively. In this situation, \mathcal{U}_{d_X} is the same as the uniformity induced on X by \mathcal{U}_{d_Y} on Y and the mapping f as before, because of (13.12). Note that (13.9) is a semi-ultrametric on X when $d_Y(\cdot, \cdot)$ is a semi-ultrametric on Y .

14 Sub-bases

Let X be a set, let \mathcal{U} be a uniformity on X , and let \mathcal{B}_0 be a subcollection of \mathcal{U} . Consider the collection \mathcal{B} of subsets of $X \times X$ that can be expressed as the intersection of finitely many elements of \mathcal{B}_0 . If \mathcal{B} is a base for \mathcal{U} , as in Section 3, then \mathcal{B}_0 is said to be a *sub-base* for \mathcal{U} . Note that $\mathcal{B}_0 \neq \emptyset$ in this case, because $\mathcal{U} \neq \emptyset$, and hence $\mathcal{B} \neq \emptyset$. If $U \in \mathcal{B}_0$, then (3.1) holds, since $U \in \mathcal{U}$. There should also be a $V_0 \in \mathcal{B}$ that satisfies (3.10), because $U \in \mathcal{B}$. Similarly, there should be a $V \in \mathcal{B}$ that satisfies (3.3).

Now let \mathcal{B}_0 be any nonempty collection of subsets of $X \times X$, and let \mathcal{B} be the collection of subsets of $X \times X$ that can be expressed as the intersection of finitely many elements of \mathcal{B}_0 , as before. If \mathcal{B}_0 satisfies the three conditions with respect to \mathcal{B} mentioned at the end of the preceding paragraph, then it is easy to see that \mathcal{B} has the analogous properties with respect to itself. Of course, the intersection of any two elements of \mathcal{B} is automatically an element of \mathcal{B} , by

construction. As in Section 3, it follows that (3.12) is a uniformity on X , and that \mathcal{B} is a base for this uniformity. Thus \mathcal{B}_0 is a sub-base for this uniformity.

Let I be a nonempty set, and suppose that \mathcal{B}_i is a sub-base for a uniformity on X for each $i \in I$. Under these conditions, one can check that

$$(14.1) \quad \bigcup_{i \in I} \mathcal{B}_i$$

satisfies the requirements of a sub-base for a uniformity on X described in the previous paragraph. Note that (14.1) is not necessarily a base for a uniformity on X , even when \mathcal{B}_i is a base for a uniformity on X for each $i \in I$. As a basic scenario, let (Y_i, \mathcal{V}_i) be a uniform space for each $i \in I$, and let f_i be a mapping from X into Y_i . This leads to a mapping $f_{i,2}$ from $X \times X$ into $Y_i \times Y_i$ for each $i \in I$, as in (9.1). We also get a base

$$(14.2) \quad \mathcal{B}_i = \{f_{i,2}^{-1}(V_i) : V_i \in \mathcal{V}_i\}$$

for a uniformity on X for each $i \in I$, as in (13.6). Thus (14.1) is a sub-base for a uniformity \mathcal{U} on X , as before. By construction, for each $i \in I$, f_i is uniformly continuous as a mapping from X into Y_i with respect to the uniformities \mathcal{U} and \mathcal{V}_i , respectively.

Let (X, \mathcal{U}) and (Y, \mathcal{V}) be uniform spaces, and let f be a mapping from X into Y . In order to check that f is uniformly continuous, it suffices to verify that (9.2) holds for every V in a sub-base for \mathcal{V} , as mentioned in Section 9. In particular, suppose that there is a sub-base for \mathcal{V} which is the union of a family of sub-bases for other uniformities on Y , as in the preceding paragraph. If f is uniformly continuous as a mapping into Y with respect to each of these other uniformities on Y , then it follows that f is uniformly continuous as a mapping into Y with respect to \mathcal{V} . The converse is trivial, since the uniform continuity of f as a mapping into Y with respect to \mathcal{V} automatically implies that f is uniformly continuous as a mapping into Y with respect to any uniformity contained in \mathcal{V} .

15 Cartesian products

Let I be a nonempty set, and suppose that (Y_i, \mathcal{V}_i) is a uniform space for each $i \in I$. Also let

$$(15.1) \quad Y = \prod_{i \in I} Y_i$$

be the Cartesian product of the Y_i 's, and let p_j be the standard coordinate projection from Y onto Y_j for each $j \in I$. This leads to a mapping $p_{j,2}$ from $Y \times Y$ onto $Y_j \times Y_j$ for each $j \in I$, as in (9.1). Put

$$(15.2) \quad \mathcal{B}_j = \{p_{j,2}^{-1}(V_j) : V_j \in \mathcal{V}_j\}$$

for each $j \in I$, which is a base for a uniformity on Y , as in (13.6). Thus

$$(15.3) \quad \bigcup_{j \in I} \mathcal{B}_j$$

is a sub-base for a uniformity \mathcal{V} on Y , as in the previous section. This uniformity \mathcal{V} is the *product uniformity* on Y associated to the given uniformities on the Y_j 's. By construction, p_j is uniformly continuous as a mapping from Y onto Y_j for each $j \in I$, with respect to the product uniformity on Y and \mathcal{V}_j on Y_j .

Let (X, \mathcal{U}) be another uniform space, and let f_i be a mapping from X into Y_i for each $i \in I$. This leads to a mapping f from X into Y , with

$$(15.4) \quad f_i = p_i \circ f$$

for each $i \in I$. If f_i is uniformly continuous for each i , then one can check that f is uniformly continuous with respect to the product uniformity on Y . More precisely, it suffices to verify that (9.2) holds for every V in the sub-base (15.3) for the product uniformity, which reduces to the uniform continuity of the f_j 's. Conversely, if f is uniformly continuous, then f_i is uniformly continuous for each $i \in I$, because it is the composition of uniformly continuous mappings, as in (15.4).

Of course,

$$(15.5) \quad Y \times Y = \left(\prod_{i \in I} Y_i \right) \times \left(\prod_{i \in I} Y_i \right)$$

can be identified with

$$(15.6) \quad \prod_{i \in I} (Y_i \times Y_i)$$

in a natural way. Let $j \in I$ be given, and observe that $p_{j,2}$ corresponds to the standard coordinate projection from (15.6) onto $Y_j \times Y_j$. If $V_j \subseteq Y_j \times Y_j$, then

$$(15.7) \quad p_{j,2}^{-1}(V_j) \subseteq Y \times Y$$

corresponds to the subset of (15.6) which is the Cartesian product of V_j with $Y_i \times Y_i$ for each $i \in I$ different from j . If $y \in Y$, then $V_j[p_j(y)]$ can be defined as a subset of Y_j as in (2.16), and similarly $(p_{j,2}^{-1}(V_j))[y]$ can be defined as a subset of Y . It is easy to see that

$$(15.8) \quad (p_{j,2}^{-1}(V_j))[y] = p_j^{-1}(V_j[p_j(y)]),$$

which is the same as the Cartesian product of $V_j[p_j(y)]$ with Y_i for each $i \in I$ with $i \neq j$.

Each Y_i has a topology associated to the given uniformity \mathcal{V}_i , as in Section 4, and there is a topology on Y associated to the product uniformity as well. There is also the product topology on Y corresponding to the topologies on the Y_i 's just mentioned. It is easy to see that an open subset of Y with respect to the product topology is an open set with respect to the topology associated to the product uniformity, directly from the definitions. To show that an open subset of Y with respect to the topology associated to the product uniformity is an open set with respect to the product topology, one can use (4.13). One can look at this in terms of local bases and sub-bases for the relevant topologies, as on p182-3 of [12].

16 Cartesian products, continued

Let (X, \mathcal{U}) be a uniform space, and let us consider the corresponding product uniformity on $X \times X$, as in the previous section. If $U, V \in \mathcal{U}$, then

$$(16.1) \quad U_1 = \{((x_1, x_2), (x'_1, x'_2)) \in (X \times X) \times (X \times X) : (x_1, x'_1) \in U\}$$

and

$$(16.2) \quad V_2 = \{((x_1, x_2), (x'_1, x'_2)) \in (X \times X) \times (X \times X) : (x_2, x'_2) \in V\}$$

are sub-basic elements of the product uniformity on $X \times X$. More precisely, U_1 and V_2 correspond to the first and second coordinate projections from $X \times X$ onto X , as in (15.2) and (15.7). Thus

$$(16.3) \quad U_1 \cap V_2 = \{((x_1, x_2), (x'_1, x'_2)) \in (X \times X) \times (X \times X) : (x_1, x'_1) \in U, (x_2, x'_2) \in V\}$$

is an element of the product uniformity on $X \times X$. The collection of these sets forms a base for the product uniformity on $X \times X$.

If $(x_1, x_2) \in X \times X$, then $U_1[(x_1, x_2)]$ and $V_2[(x_1, x_2)]$ can be defined as subsets of $X \times X$ as in (2.16), and we have that

$$(16.4) \quad U_1[(x_1, x_2)] = (U[x_1]) \times X$$

and

$$(16.5) \quad V_2[(x_1, x_2)] = X \times (V[x_2]).$$

Similarly,

$$(16.6) \quad (U_1 \cap V_2)[(x_1, x_2)] = (U[x_1]) \times (V[x_2]).$$

If $A \subseteq X \times X$, then $(U_1 \cap V_2)[A]$ can be defined as a subset of $X \times X$ as in (2.13), which is equivalent to (2.17). In this situation, (2.17) reduces to

$$(16.7) \quad (U_1 \cap V_2)[A] = \bigcup_{(x_1, x_2) \in A} (U[x_1]) \times (V[x_2]),$$

because of (16.6).

If $A = \Delta$, the diagonal in $X \times X$, then (16.7) reduces to

$$(16.8) \quad (U_1 \cap V_2)[\Delta] = \bigcup_{x \in X} (U[x]) \times (V[x]).$$

It follows that

$$(16.9) \quad (U_1 \cap V_2)[\Delta] = \tilde{U} * V,$$

where \tilde{U} is as defined in (2.2), and then $\tilde{U} * V$ is defined as in (2.3). If W is any element of \mathcal{U} , then there are $U, V \in \mathcal{U}$ such that

$$(16.10) \quad \tilde{U} * V \subseteq W.$$

More precisely, there is a $V \in \mathcal{U}$ such that $V * V \subseteq W$, as in (3.3), and one can take $U = \tilde{V}$.

17 Compatible semimetrics

Let (X, \mathcal{U}) be a uniform space again, and let $d(\cdot, \cdot)$ be a semimetric on X . Also let \mathcal{U}_d be the uniformity on X associated to $d(\cdot, \cdot)$ as in Section 3. Let us say that $d(\cdot, \cdot)$ is *compatible* with \mathcal{U} if

$$(17.1) \quad \mathcal{U}_d \subseteq \mathcal{U}.$$

Equivalently, this means that the identity mapping on X is uniformly continuous as a mapping from X equipped with \mathcal{U} into X equipped with \mathcal{U}_d . If $U_d(r)$ is defined as in (2.6) for each $r > 0$, then (17.1) is the same as saying that

$$(17.2) \quad U_d(r) \in \mathcal{U}$$

for every $r > 0$.

As in Theorem 11 on p183 of [12], $d(\cdot, \cdot)$ is compatible with \mathcal{U} in the sense described in the preceding paragraph if and only if $d(\cdot, \cdot)$ is uniformly continuous on $X \times X$ with respect to the product uniformity corresponding to \mathcal{U} on X as in the previous sections. Here we use the standard uniform structure on the real line, corresponding to the standard Euclidean metric, for the range of $d(\cdot, \cdot)$ on $X \times X$. To prove the “if” part, suppose that $d(\cdot, \cdot)$ is uniformly continuous on $X \times X$, and let $r > 0$. Under these conditions, there are $U, V \in \mathcal{U}$ so that if U_1 and V_2 are as in (16.4) and (16.5), respectively, then

$$(17.3) \quad |d(x_1, x_2) - d(x'_1, x'_2)| < r$$

for every $((x_1, x_2), (x'_1, x'_2)) \in U_1 \cap V_2$. This uses the fact that the sets $U_1 \cap V_2$ with $U, V \in \mathcal{U}$ form a base for the product uniformity on $X \times X$, as in the previous section. If we restrict our attention to $x_1 = x_2$ in (17.3), then we get that

$$(17.4) \quad d(x'_1, x'_2) < r$$

for every $(x'_1, x'_2) \in X \times X$ for which there is an $x \in X$ such that

$$(17.5) \quad ((x, x), (x'_1, x'_2)) \in U_1 \cap V_2.$$

Equivalently, this means that (17.4) holds when (x'_1, x'_2) is an element of (16.8), so that

$$(17.6) \quad (U_1 \cap V_2)[\Delta] \subseteq U_d(r),$$

where $U_d(r)$ is as in (2.6). It follows that

$$(17.7) \quad \tilde{U} * V \subseteq U_d(r),$$

using also (16.9). Remember that

$$(17.8) \quad \Delta \subseteq U, V,$$

because $U, V \in \mathcal{U}$, and hence $\Delta \subseteq \tilde{U}$ as well. Thus (17.7) implies that

$$(17.9) \quad \tilde{U}, V \subseteq U_d(r),$$

which corresponds to taking x equal to x'_1 or x'_2 in (17.5). Note that (17.9) also implies that $U \subseteq U_d(r)$, because $U_d(r)$ is symmetric. It follows that (17.2) holds, as desired, by the definition of a uniformity. More precisely, this shows that $d(\cdot, \cdot)$ is compatible with \mathcal{U} on X when $d(\cdot, \cdot)$ is uniformly continuous along the diagonal Δ in $X \times X$, which corresponds to the restriction to $x_1 = x_2$ in (17.3). Of course, if $d(\cdot, \cdot)$ is uniformly continuous on $X \times X$, then $d(\cdot, \cdot)$ is uniformly continuous along any subset of $X \times X$.

Before proving the converse, let us record some simple estimates. Using the triangle inequality twice, we get that

$$(17.10) \quad d(x_1, x_2) \leq d(x'_1, x'_2) + d(x_1, x'_1) + d(x_2, x'_2)$$

for every $x_1, x_2, x'_1, x'_2 \in X$, and hence

$$(17.11) \quad d(x_1, x_2) - d(x'_1, x'_2) \leq d(x_1, x'_1) + d(x_2, x'_2).$$

Similarly,

$$(17.12) \quad d(x'_1, x'_2) - d(x_1, x_2) \leq d(x_1, x'_1) + d(x_2, x'_2).$$

Combining (17.11) and (17.12), we get that

$$(17.13) \quad |d(x_1, x_2) - d(x'_1, x'_2)| \leq d(x_1, x'_1) + d(x_2, x'_2).$$

If $(x_1, x'_1), (x_2, x'_2) \in U_d(r)$ for some $r > 0$, then it follows that

$$(17.14) \quad |d(x_1, x_2) - d(x'_1, x'_2)| < r + r = 2r,$$

by the definition (2.6) of $U_d(r)$.

Suppose now that $d(\cdot, \cdot)$ is compatible with \mathcal{U} on X , and let $r > 0$ be given. Thus (17.2) holds, and we can take $U = V = U_d(r)$ in (16.3), to get an element of the product uniformity on $X \times X$ associated to \mathcal{U} on X . In this case, we have just seen that (17.14) holds when $((x_1, x_2), (x'_1, x'_2))$ is an element of (16.3). This implies that $d(\cdot, \cdot)$ is uniformly continuous on $X \times X$ with respect to the product uniformity associated to \mathcal{U} on X , as desired. In particular, $d(\cdot, \cdot)$ is uniformly continuous on $X \times X$ with respect to the product uniformity associated to the uniformity \mathcal{U}_d determined on X by itself, as in Section 3.

18 Collections of semimetrics

Let X be a set, and let \mathcal{M} be a nonempty collection of semimetrics on X . Also let $U_d(r)$ be the subset of $X \times X$ associated to $d \in \mathcal{M}$ and a positive real number r as in (2.6). Thus

$$(18.1) \quad \mathcal{B}_d = \{U_d(r) : r > 0\}$$

is a base for a uniformity \mathcal{U}_d on X for each $d \in \mathcal{M}$, as discussed in Section 3. It follows that

$$(18.2) \quad \bigcup_{d \in \mathcal{M}} \mathcal{B}_d$$

is a sub-base for a uniformity on X , as in Section 14. Of course, each element of \mathcal{M} is compatible with this uniformity on X , by construction.

Now let I be nonempty set, let Y_i be a set for each $i \in I$, and let

$$(18.3) \quad Y = \prod_{i \in I} Y_i$$

be the corresponding Cartesian product, as in Section 15. Suppose that \mathcal{M}_j is a nonempty collection of semimetrics on Y_j for each $j \in I$, and let \mathcal{V}_j be the uniformity on Y_j associated to \mathcal{M}_j as in the preceding paragraph. If $j \in I$ and $d_j \in \mathcal{M}_j$, then put

$$(18.4) \quad \widehat{d}_j(y, y') = d_j(p_j(y), p_j(y'))$$

for every $y, y' \in Y$, where p_j is the standard coordinate projection from Y onto Y_j , as before. Note that (18.4) defines a semimetric on Y , as in (13.9). Let $\widehat{\mathcal{M}}_j$ be the collection of semimetrics on Y of the form (18.4) with $d_j(\cdot, \cdot) \in \mathcal{M}_j$, and put

$$(18.5) \quad \widehat{\mathcal{M}} = \bigcup_{j \in I} \widehat{\mathcal{M}}_j.$$

This is a nonempty collection of semimetrics on Y , which determines a uniform structure on Y as in the previous paragraph. In this situation, the uniform structure on Y associated to $\widehat{\mathcal{M}}$ is the same as the product uniformity on Y corresponding to the uniformity \mathcal{V}_j on Y_j associated to \mathcal{M}_j for each $j \in I$.

Suppose that d_1, \dots, d_n are finitely many semimetrics on a set X . Under these conditions, one can check that

$$(18.6) \quad d(x, y) = \max_{1 \leq j \leq n} d_j(x, y)$$

defines a semimetric on X as well. Put

$$(18.7) \quad U_{d_j}(r) = \{(x, y) \in X \times X : d_j(x, y) < r\}$$

for each $j = 1, \dots, n$ and $r > 0$, and let $U_d(r)$ be the analogous set corresponding to d , as in (2.6). Observe that

$$(18.8) \quad U_d(r) = \bigcap_{j=1}^n U_{d_j}(r)$$

for every $r > 0$. Let \mathcal{B}_d be the collection of subsets of $X \times X$ of the form $U_d(r)$ for some $r > 0$, as in Section 3, and let \mathcal{B}_{d_j} be the analogous collection associated to d_j for each $j = 1, \dots, n$. Thus \mathcal{B}_d is a base for the uniformity \mathcal{U}_d on X associated to d , as before, and similarly \mathcal{B}_{d_j} is a base for the uniformity \mathcal{U}_{d_j} on X associated to d_j for each $j = 1, \dots, n$. We also have that

$$(18.9) \quad \bigcup_{j=1}^n \mathcal{B}_{d_j}$$

is a sub-base for a uniformity on X , as in (18.2). One can check that this is actually a sub-base for \mathcal{U}_d , using (18.8). If d_1, \dots, d_n are semi-ultrametrics on X , then (18.6) is a semi-ultrametric on X too.

If (X, \mathcal{U}) is any uniform space, then it is well known that there is a collection \mathcal{M} of semimetrics on X for which \mathcal{U} is the corresponding uniformity.

19 Sequences of semimetrics

Let $d(\cdot, \cdot)$ be a semimetric on a set X , and let t be a positive real number. Under these conditions, it is easy to see that

$$(19.1) \quad d_t(x, y) = \min(d(x, y), t)$$

also defines a semimetric on X . Put

$$(19.2) \quad U_{d_t}(r) = \{(x, y) \in X \times X : d_t(x, y) < r\}$$

for each $r > 0$, and let $U_d(r)$ be the analogous subset of $X \times X$ corresponding to $d(\cdot, \cdot)$, as in (2.6). Observe that

$$(19.3) \quad U_{d_t}(r) = U_d(r)$$

when $r \leq t$, and

$$(19.4) \quad U_{d_t}(r) = X \times X$$

when $r > t$. It follows that the uniformity on X associated to d_t as in Section 3 is the same as the uniformity associated to d for every $t > 0$. In particular, the corresponding topologies are the same. Note that (19.1) is a semi-ultrametric on X when $d(\cdot, \cdot)$ is a semi-ultrametric on X .

Now let d_1, d_2, d_3, \dots be an infinite sequence of semimetrics on X , and put

$$(19.5) \quad d'_j(x, y) = \min(d_j(x, y), 1/j)$$

for every positive integer j and $x, y \in X$. Thus d'_j is also a semimetric on X for each j , as in the previous paragraph, and d'_j determines the same uniformity on X as d_j . Put

$$(19.6) \quad d'(x, y) = \max_{j \geq 1} d'_j(x, y)$$

for each $x, y \in X$. Of course, (19.6) is equal to 0 when $d'_j(x, y) = 0$ for every j . Otherwise, if $d'_l(x, y) > 0$ for some l , then (19.6) reduces to the maximum over finitely many j , because (19.5) is automatically less than or equal to $1/j$. This implies that the maximum in (19.6) is always attained, and one can check that (19.6) defines a semimetric on X . If d_j is a semi-ultrametric on X for each j , then d'_j is a semi-ultrametric on X for each j , and (19.6) is a semi-ultrametric on X as well.

As usual, for each $r > 0$, we put

$$(19.7) \quad U_{d_j}(r) = \{(x, y) \in X \times X : d_j(x, y) < r\},$$

$$(19.8) \quad U_{d'_j}(r) = \{(x, y) \in X \times X : d'_j(x, y) < r\},$$

and

$$(19.9) \quad U_{d'}(r) = \{(x, y) \in X \times X : d'(x, y) < r\}.$$

We also have that

$$(19.10) \quad \begin{aligned} U_{d'_j}(r) &= U_{d_j}(r) && \text{when } r \leq 1/j \\ &= X \times X && \text{when } r > 1/j, \end{aligned}$$

as in (19.3) and (19.4). By construction,

$$(19.11) \quad U_{d'}(r) = \bigcap_{j=1}^{\infty} U_{d'_j}(r)$$

for every $r > 0$. Combining this with (19.10), we get that

$$(19.12) \quad U_{d'}(r) = \bigcap_{j=1}^{[1/r]} U_{d_j}(r)$$

when $0 < r \leq 1$, where $[1/r]$ is the largest positive integer less than or equal to $1/r$. If $r > 1$, then

$$(19.13) \quad U_{d'}(r) = X \times X,$$

because (19.6) is always less than or equal to 1. Using (19.12), one can check that the uniformity on X associated to d' as in Section 3 is the same as the uniformity on X associated to the sequence of semimetrics d'_1, d'_2, d'_3, \dots , as in the previous section. This is also the same as the uniformity on X associated to the initial sequence of semimetrics d_1, d_2, d_3, \dots , since the uniformities on X associated to d_j and to d'_j are the same for each j , as before.

Let $d(\cdot, \cdot)$ be any semimetric on X again, and let \mathcal{U}_d be the corresponding uniformity on X , as in Section 3. We have seen that the collection of subsets of $X \times X$ of the form $U_d(r)$ as in (2.6) for some $r > 0$ forms a base for \mathcal{U}_d . More precisely, one can get a base for \mathcal{U}_r using the sets $U_d(r)$ corresponding to a sequence of positive real numbers r converging to 0. In particular, there is a base for \mathcal{U} consisting of finitely or countably many subsets of $X \times X$. Conversely, if \mathcal{U} is a uniformity on X , and if there is a base for \mathcal{U} with only finitely or countably many elements, then it is well known that there is a semimetric on X for which \mathcal{U} is the associated uniformity, as on p186 of [12].

20 Collections of semi-ultrametrics

Let X be a set, and let U be a subset of $X \times X$. Thus U corresponds to a binary relation $x \sim y$ on X , as in Section 2. Of course, $x \sim y$ is reflexive on X if and only if U contains the diagonal Δ in (2.1) as a subset. Similarly, $x \sim y$ is symmetric on X if and only if U is symmetric in the sense that \tilde{U} in (2.2) is equal to U . Transitivity of $x \sim y$ on X may be expressed as

$$(20.1) \quad U * U \subseteq U,$$

using the notation in (2.3). If $x \sim y$ is reflexive on X , so that $\Delta \subseteq X$, then we have that

$$(20.2) \quad U \subseteq U * U,$$

by (2.4). It follows that

$$(20.3) \quad U * U = U$$

when $x \sim y$ is both transitive and reflexive on X .

Let U_1, U_2 be subsets of $X \times X$, which correspond to binary relations $x \sim_1 y$ and $x \sim_2 y$ on X , respectively, as before. Note that $U_1 \cap U_2$ corresponds to the binary relation $x \sim y$ defined on X by requiring that both $x \sim_1 y$ and $x \sim_2 y$ hold. If $x \sim_1 y$ and $x \sim_2 y$ are equivalence relations on X , then $x \sim y$ is an equivalence relation on X as well.

Let \mathcal{B}_0 be a nonempty collection of subsets of $X \times X$, each of which corresponds to an equivalence relation on X . Also let \mathcal{B} be the collection of subsets of $X \times X$ that can be expressed as the intersection of finitely many elements of \mathcal{B}_0 . Every element of \mathcal{B} corresponds to an equivalence relation on X too, by the remarks in the previous paragraph. This means that for each $U \in \mathcal{B}$, we have that $\Delta \subseteq U$, $\tilde{U} = U$, and U satisfies (20.1). Using this, it is easy to see that \mathcal{B} is a base for a uniformity on X , since \mathcal{B} is automatically closed under finite intersections.

Suppose that \mathcal{M} is a nonempty collection of semi-ultrametrics on X , and let \mathcal{U} be the corresponding uniformity on X , as in Section 18. Also let $U_d(r)$ be the subset of $X \times X$ associated to $d \in \mathcal{M}$ and $r > 0$ as in (2.6). Note that $U_d(r)$ corresponds to an equivalence relation on X for every $d \in \mathcal{M}$ and $r > 0$, as in Section 8. By construction, the collection of subsets of $X \times X$ of the form $U_d(r)$ for some $d \in \mathcal{M}$ and $r > 0$ is a sub-base for the uniformity \mathcal{U} on X associated to \mathcal{M} .

Let $x \sim y$ be any equivalence relation on X , and let $d(x, y)$ be the corresponding discrete semi-ultrametric on X , as in (8.8). Also let $U_d(r)$ be the subset of $X \times X$ corresponding to $d(x, y)$ and $r > 0$ as in (2.6). Observe that

$$(20.4) \quad U_d(r) = \{(x, y) \in X \times X : x \sim y\}$$

when $0 < r \leq 1$, and $U_d(r) = X \times X$ when $r > 1$.

Let us now return to the situation where \mathcal{B}_0 is a nonempty collection of subsets of $X \times X$, each of which corresponds to an equivalence relation on X . The earlier remarks imply that \mathcal{B}_0 is a sub-base for a uniformity \mathcal{U} on X . Let \mathcal{M}_0 be the collection of discrete semi-ultrametrics on X that correspond to equivalence relations on X associated to elements of \mathcal{B}_0 , as in (8.8). Under these conditions, \mathcal{U} is the same as the uniformity on X associated to \mathcal{M}_0 as in Section 18.

If \mathcal{B}_0 has only finitely many elements, then \mathcal{M}_0 has only finitely many elements too. In this case, the same uniformity on X is determined by a single semi-ultrametric, as in Section 18. Equivalently, one can take the intersection of the elements of \mathcal{B}_0 , to get another subset of $X \times X$ that corresponds to an equivalence relation on X . The discrete semi-ultrametric on X associated to that subset of $X \times X$ determines the same uniformity on X .

If \mathcal{B}_0 is countably infinite, then \mathcal{M}_0 is countably infinite as well. In this case, one can get a semi-ultrametric on X that determines the same uniformity on X as in the previous section. Conversely, let $d(\cdot, \cdot)$ be any semi-ultrametric on X , and let \mathcal{U}_d be the corresponding uniformity on X , as in Section 3. If $U_d(r)$ is as in (2.6), then $U_d(r)$ corresponds to an equivalence relation on X for every $r > 0$, as in Section 8. As in the previous section, one can get a base for \mathcal{U}_d consisting of the sets $U_d(r)$ for a sequence of positive real numbers r converging to 0.

21 q -Semimetrics

Let X be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined on $X \times X$ is said to be a q -semimetric if it satisfies (1.1), (1.2), and

$$(21.1) \quad d(x, z)^q \leq d(x, y)^q + d(y, z)^q$$

for every $x, y, z \in X$, instead of the ordinary triangle inequality (1.3). This is the same as saying that $d(x, y)^q$ is an ordinary semimetric on X . If $d(x, y)$ also satisfies (1.4), then it is said to be a q -metric on X . Of course, (21.1) is equivalent to asking that

$$(21.2) \quad d(x, z) \leq (d(x, y)^q + d(y, z)^q)^{1/q}$$

for every $x, y, z \in X$.

Let $d(x, y)$ be a q -semimetric on X for some $q > 0$, and let $U_d(r) \subseteq X \times X$ be as in (2.6) for each $r > 0$. This satisfies (2.7), (2.8), (2.10), and (2.11), as in the case of ordinary semimetrics. Instead of (2.9), we have that

$$(21.3) \quad U_d(r_1) * U_d(r_2) \subseteq U_d((r_1^q + r_2^q)^{1/q})$$

for every $r_1, r_2 > 0$, by (21.2). Using this, one can define a uniformity \mathcal{U}_d on X associated to $d(x, y)$ in the same way as for ordinary semimetrics, as in Section 3. If

$$(21.4) \quad U_{d^q}(r) = \{(x, y) \in X \times X : d(x, y)^q < r\}$$

is the analogous set associated to the semimetric $d(x, y)^q$ on X , then

$$(21.5) \quad U_{d^q}(r^q) = U_d(r)$$

for every $r > 0$. This implies that \mathcal{U}_d is the same as the uniformity \mathcal{U}_{d^q} on X associated to $d(x, y)^q$ as in Section 3. In particular, the corresponding topologies on X are the same.

Let \mathcal{M} be a nonempty collection of q -semimetrics on X , where one can let $q > 0$ depend on the element of \mathcal{M} . One can define a uniformity on X using \mathcal{M} in the same way as in Section 18. Alternatively, one can get a collection of ordinary semimetrics on X by replacing each $d \in \mathcal{M}$ with $d(x, y)^q$ for a suitable $q > 0$. Such a collection of ordinary semimetrics leads to the same uniformity on X , because of (21.5).

If a, b are nonnegative real numbers, then

$$(21.6) \quad \max(a, b) \leq (a^q + b^q)^{1/q} \leq 2^{1/q} \max(a, b)$$

for every $q > 0$. If $0 < q \leq q' < \infty$, then we get that

$$(21.7) \quad a^{q'} + b^{q'} \leq \max(a, b)^{q'-q} (a^q + b^q) \leq (a^q + b^q)^{(q'-q)/q+1} = (a^q + b^q)^{q'/q}.$$

Hence

$$(21.8) \quad (a^{q'} + b^{q'})^{1/q'} \leq (a^q + b^q)^{1/q}.$$

It follows that every q' -semimetric on X is a q -semimetric on X when $q \leq q'$, using also (21.2). Note that a semi-ultrametric on X is a q -semimetric for each $q > 0$, because of the first inequality in (21.6). Using both inequalities in (21.6), we have that

$$(21.9) \quad \lim_{q \rightarrow \infty} (a^q + b^q)^{1/q} = \max(a, b)$$

for every $a, b \geq 0$. Thus one can think of a semi-ultrametric as being a q -semimetric with $q = \infty$.

22 q -Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|\cdot|$ on k is said to be an q -absolute value function on k if it satisfies the following three conditions. First,

$$(22.1) \quad |x| = 0 \text{ if and only if } x = 0.$$

Second,

$$(22.2) \quad |xy| = |x| |y|$$

for every $x, y \in k$. Third,

$$(22.3) \quad |x + y|^q \leq |x|^q + |y|^q$$

for every $x, y \in k$. As before, (22.3) is equivalent to asking that

$$(22.4) \quad |x + y| \leq (|x|^q + |y|^q)^{1/q}$$

for every $x, y \in k$. A q -absolute value function on k with $q = 1$ is also known simply as an absolute value function on k . Thus $|x|$ is a q -absolute value function on k if and only if $|x|^q$ is an absolute value function on k . Of course, the standard absolute value functions on the fields \mathbf{R} and \mathbf{C} of real and complex numbers are absolute value functions in this sense. If $0 < q \leq q' < \infty$ and $|\cdot|$ is a q' -absolute value function on a field k , then $|\cdot|$ is a q -absolute value function on k too, by (21.8).

Similarly, a nonnegative real-valued function $|\cdot|$ on a field k is said to be an *ultrametric absolute value function* on k if it satisfies (22.1), (22.2), and

$$(22.5) \quad |x + y| \leq \max(|x|, |y|)$$

for every $x, y \in k$. An ultrametric absolute value function on k is a q -absolute value function for every $q > 0$, by the first inequality in (21.6). One can also think of an ultrametric absolute value function on k as being a q -absolute value function with $q = \infty$, because of (21.9). The *trivial absolute value function* is defined on any field k by putting $|0| = 0$ and

$$(22.6) \quad |x| = 1$$

for every $x \in k$ with $x \neq 0$. It is easy to see that this is an ultrametric absolute value function on k .

Suppose that $|\cdot|$ is a nonnegative real-valued function on a field k that satisfies (22.1) and (22.2). If 1 is the multiplicative identity element in k , then $1 \neq 0$ in k , by the definition of a field, and hence $|1| > 0$. We also have that $1^2 = 1$ in k , so that $|1| = |1^2| = |1|^2$, and hence

$$(22.7) \quad |1| = 1.$$

If $x \in k$ satisfies $x^n = 1$ for some positive integer n , then

$$(22.8) \quad |x|^n = |x^n| = |1| = 1,$$

so that $|x| = 1$. In particular,

$$(22.9) \quad |-1| = 1,$$

which implies that

$$(22.10) \quad |-x| = |(-1)x| = |-1||x| = |x|$$

for every $x \in k$.

If $|\cdot|$ is a q -absolute value function on k for some $q > 0$, then it follows that

$$(22.11) \quad d(x, y) = |x - y|$$

defines a q -metric on k . Similarly, if $|\cdot|$ is an ultrametric absolute value function on k , then (22.11) is an ultrametric on k . The ultrametric corresponding to the trivial absolute value function on k as in (22.11) is the discrete metric.

23 q -Seminorms

Let k be a field, let $|\cdot|$ be a q -absolute value function on k for some $q > 0$, and let V be a vector space over k . A nonnegative real-valued function N on V is said to be a q -*seminorm* on V if

$$(23.1) \quad N(tv) = |t| N(v)$$

for every $v \in V$ and $t \in k$, and

$$(23.2) \quad N(v + w)^q \leq N(v)^q + N(w)^q$$

for every $v, w \in V$. Of course, (23.1) implies that $N(0) = 0$, by taking $t = 0$. If we also have that

$$(23.3) \quad N(v) > 0$$

for every $v \in V$ with $v \neq 0$, then N is said to be a q -norm on V .

If $q = 1$, then a q -seminorm may simply be called a *seminorm*, and a q -norm may be called a *norm*. Remember that $|\cdot|$ is a q -absolute value function on k if and only if $|\cdot|^q$ is an absolute value function on k . Similarly, N is a q -seminorm on V with respect to $|\cdot|$ on k if and only if N^q is a seminorm on V with respect to $|\cdot|^q$ on k . Of course, there is an analogous statement for q -norms.

As usual, (23.2) is the same as asking that

$$(23.4) \quad N(v + w) \leq (N(v)^q + N(w)^q)^{1/q}$$

for every $v, w \in V$. Suppose for the moment that $0 < q \leq q' \leq \infty$, and that $|\cdot|$ is a q' -absolute value function on k . This implies that $|\cdot|$ is a q -absolute value function on k as well, as in the previous section. Similarly, if N is a q' -seminorm on V , then N is a q -seminorm on V , because of (21.8). The analogous statement for q -norms follows by including the additional condition (23.3).

Suppose now that $|\cdot|$ is an ultrametric absolute value function on k . A nonnegative real-valued function N is said to be a *semi-ultranorm* on V if N satisfies (23.1) and

$$(23.5) \quad N(v + w) \leq \max(N(v), N(w))$$

for every $v, w \in V$. If N also satisfies (23.3), then N is said to be an *ultranorm* on V . Note that (23.5) implies (23.4) for every $q > 0$, by the first inequality in (21.6). Thus a semi-ultranorm on V is a q -seminorm on V for every $q > 0$, and similarly for ultranorms. This implicitly uses the fact that an ultrametric absolute value function on k is a q -absolute value function on k for every $q > 0$, as in the previous section. One can think of semi-ultranorms and ultranorms as being the $q = \infty$ versions of q -seminorms and q -norms, respectively, because of (21.9).

Let $|\cdot|$ be a q -absolute value function on k for some $q > 0$ again. If N is a q -seminorm on V , then

$$(23.6) \quad d(v, w) = N(v - w)$$

is a q -semimetric on V . If N is a q -norm on V , then (23.6) is a q -metric on V . Similarly, if $|\cdot|$ is an ultrametric absolute value function on k , and if N is a semi-ultranorm on V , then (23.6) is a semi-ultrametric on V . Under the same conditions, if N is an ultranorm on V , then (23.6) is an ultrametric on V .

Let $|\cdot|$ be the trivial absolute value function on k , which is an ultrametric absolute value function on k , as in the previous section. The *trivial ultranorm* is defined by putting $N(0) = 0$ and

$$(23.7) \quad N(v) = 1$$

for every $v \in V$ with $v \neq 0$. It is easy to see that this is an ultranorm on V , for which the corresponding metric as in (23.6) is the same as the discrete metric.

We have included the requirement that $|\cdot|$ be a q -absolute value function on k in the definition of a q -seminorm for convenience, and to avoid trivialities. If $N(v) > 0$ for some $v \in V$, then (23.1) and (23.2) imply (22.3), as well as (22.2). Similarly, (23.1) and (23.5) imply (22.5) in this case.

Part II

Connectedness and dimension 0

24 Connected sets

As usual, a pair A, B of subsets of a topological space X are said to be *separated* in X if

$$(24.1) \quad \overline{A} \cap B = A \cap \overline{B} = \emptyset,$$

where $\overline{A}, \overline{B}$ are the closures of A, B in X , respectively. In particular, disjoint closed subsets of X are automatically separated, and one can check that disjoint open subsets of X are separated as well. A set $E \subseteq X$ is said to be *connected* if it cannot be expressed as $A \cup B$, where A, B are nonempty separated subsets of X . Suppose that $Y \subseteq X$ is equipped with the topology induced by the given topology on X . It is well known that $A, B \subseteq Y$ are separated as subsets of Y with respect to the induced topology if and only if A and B are separated in X . This is because the closures of A, B in Y with respect to the induced topology are the same as the intersections of Y with the closures of A, B in X . It follows that $E \subseteq Y$ is connected with respect to the induced topology on Y if and only if E is connected as a subset of X .

If A, B are separated subsets of X such that

$$(24.2) \quad A \cup B = X,$$

then it is easy to see that A and B are each both open and closed as subsets of X . It follows that X is connected as a subset of itself if and only if it cannot be expressed as the union of two nonempty disjoint open sets, which is the same as saying that X cannot be expressed as the union of two nonempty disjoint closed sets. Of course, $A \subseteq X$ is both open and closed in X exactly when A and $X \setminus A$ are both open in X , which is the same as saying that they are both closed in X . Thus X is connected if and only if there are no nonempty proper subsets of X that are both open and closed. If $E \subseteq X$, then the remarks in the previous paragraph can be applied to $Y = E$, to get that E is connected as a subset of X if and only if E is connected as a subset of itself, with respect to the induced topology.

Let Z be another topological space, and suppose that f is a continuous mapping from X into Z . If E is a connected subset of X , then it is well known that $f(E)$ is connected in Z . As in the previous paragraph, one may as well suppose that $E = X$, since otherwise one can simply restrict f to E . Similarly,

one can restrict one's attention to the case where f is surjective, since otherwise one can replace Z with $f(X)$, where $f(X)$ is equipped with the topology induced by the one on Z . With these reductions, one can use the characterization of connectedness in terms of open or closed sets, as in the preceding paragraph.

Suppose that $E = A \cup B$, where A, B are separated subsets of X . Observe that

$$(24.3) \quad A = \overline{A} \cap E, \quad B = \overline{B} \cap E.$$

If E is compact in X , then it follows that A and B are compact in X too, because the intersection of a closed set and a compact set is compact. Of course, if E is not connected, then we can take A and B to be nonempty. If X is Hausdorff, then it is well known that compact subsets of X are closed in X .

25 U -Separated sets

Let X be a set, let A, B be subsets of X , and let U be a subset of $X \times X$. Suppose that

$$(25.1) \quad \Delta \subseteq U,$$

where Δ is the diagonal in $X \times X$, as in (2.1). Let us say that A, B are U -separated in X if for every $x \in A$ and $y \in B$ we have that

$$(25.2) \quad (x, y) \notin U.$$

Of course, this implies that A and B are disjoint, because of (25.1).

Equivalently, A, B are U -separated in X when

$$(25.3) \quad U \cap (A \times B) = \emptyset.$$

This is the same as saying that

$$(25.4) \quad U[A] \cap B = \emptyset,$$

where $U[A]$ is as in (2.13). We can also reformulate (25.3) as saying that

$$(25.5) \quad U \subseteq (X \times X) \setminus (A \times B).$$

Note that A, B are U -separated if and only if B, A are \tilde{U} -separated, where \tilde{U} is as in (2.2). This is the same as saying that (25.3) holds if and only if

$$(25.6) \quad \tilde{U} \cap (B \times A) = \emptyset,$$

which is equivalent to

$$(25.7) \quad \tilde{U} \subseteq (X \times X) \setminus (B \times A).$$

As before, (25.6) is also equivalent to

$$(25.8) \quad A \cap (\tilde{U}[B]) = \emptyset,$$

so that (25.4) and (25.8) are equivalent as well.

If U is symmetric, in the sense that $\widetilde{U} = U$, then the condition that A, B be U -separated is symmetric in A and B . In this case, the combination of (25.3) and (25.6) is equivalent to

$$(25.9) \quad U \cap ((A \times B) \cup (B \times A)) = \emptyset,$$

which is the same as saying that

$$(25.10) \quad U \subseteq (X \times X) \setminus ((A \times B) \cup (B \times A)).$$

Let $d(\cdot, \cdot)$ be a q -semimetric on X for some $q > 0$. We say that $A, B \subseteq X$ are r -separated in X with respect to $d(\cdot, \cdot)$ if

$$(25.11) \quad d(x, y) \geq r$$

for every $x \in A$ and $y \in B$. If $U = U_d(r)$ is as in (2.6), then this is the same as asking that A, B be U -separated, as before.

Let U be any subset of $X \times X$ that satisfies (25.1) again, and let U_1, U_2 be subsets of $X \times X$ such that

$$(25.12) \quad U_1 * \widetilde{U}_2 \subseteq U.$$

Observe that

$$(25.13) \quad \widetilde{U}_2[U_1[A]] = (U_1 * \widetilde{U}_2)[A] \subseteq U[A]$$

for every $A \subseteq X$, using (2.14) in the first step, and (25.12) in the second step. If $A, B \subseteq X$, then the equivalence between (25.4) and (25.8) implies that

$$(25.14) \quad (\widetilde{U}_2[U_1[A]]) \cap B = \emptyset$$

holds if and only if

$$(25.15) \quad (U_1[A]) \cap (U_2[B]) = \emptyset.$$

More precisely, $U_1[A]$ plays the role here that A had before, and \widetilde{U}_2 plays the role that U had before. If A, B are U -separated in X , then (25.4) and (25.13) imply that (25.14) holds, so that (25.15) holds as well. If we have that

$$(25.16) \quad U_1 * \widetilde{U}_2 = U$$

instead of (25.12), then the inclusion in the second step in (25.13) can be replaced with an equality. In this case, (25.15) implies that A, B are U -separated in X , by essentially the same argument.

Suppose that $U_{1,1}, U_{1,2} \subseteq X \times X$ satisfy

$$(25.17) \quad U_{1,1} * U_{1,2} \subseteq U_1,$$

so that

$$(25.18) \quad U_{1,2}[U_{1,1}[A]] \subseteq U_1[A],$$

as in (2.14). Under these conditions, (25.15) implies that

$$(25.19) \quad (U_{1,2}[U_{1,1}[A]]) \cap (U_2[B]) = \emptyset.$$

This means that $U_{1,1}[A]$ and $U_2[B]$ are $U_{1,2}$ -separated in X , at least if $\Delta \subseteq U_{1,2}$, so that this condition is defined.

26 Uniformly separated sets

Let (X, \mathcal{U}) be a uniform space, and let A, B be subsets of X . Let us say that A, B are *uniformly separated* in X if there is a $U \in \mathcal{U}$ such that A, B are U -separated, as in the previous section. Remember that every $U \in \mathcal{U}$ satisfies (25.1), by definition of a uniformity.

As in the previous section, A and B are U -separated in X if and only if B and A are \tilde{U} -separated in X . Thus the property of being uniformly separated in X is symmetric in A and B , because of (3.2). Of course, if A, B are U -separated for some $U \subseteq X \times X$, then A, B satisfy the analogous condition with respect to any subset of U . If $U \in \mathcal{U}$, then $U \cap \tilde{U}$ is a symmetric element of \mathcal{U} that is contained in U . Thus we may as well ask that U be symmetric in the definition of uniformly separated subsets of X .

We have seen that A and B are U -separated in X if and only if (25.5) holds. If $U \in \mathcal{U}$, then it follows that

$$(26.1) \quad (X \times X) \setminus (A \times B)$$

is an element of \mathcal{U} . Conversely, if (26.1) is an element of \mathcal{U} , then we can simply take U to be (26.1), so that A and B are automatically U -separated in X . Thus A and B are uniformly separated in X with respect to \mathcal{U} if and only if (26.1) is an element of \mathcal{U} .

Using (3.2), we get that (26.1) is an element of \mathcal{U} if and only if

$$(26.2) \quad (X \times X) \setminus (B \times A)$$

is an element of \mathcal{U} . Of course, this corresponds to the fact that the property of being uniformly separated is symmetric in A and B . In this case, we get that

$$(26.3) \quad \begin{aligned} & ((X \times X) \setminus (A \times B)) \cap ((X \times X) \setminus (B \times A)) \\ &= (X \times X) \setminus ((A \times B) \cup (B \times A)) \end{aligned}$$

is an element of \mathcal{U} as well.

Let $\overline{A}, \overline{B}$ be the closures of $A, B \subseteq X$ with respect to the topology on X associated to \mathcal{U} as in Section 4. Thus

$$(26.4) \quad \overline{A} \subseteq U[A], \quad \overline{B} \subseteq \tilde{U}[B]$$

for every $U \in \mathcal{U}$, as in Section 5. Suppose that A, B are uniformly separated in X , so that there is a $U \in \mathcal{U}$ such that (25.4) and (25.8) hold. This implies that A, B are separated in X with respect to associated topology as in (24.1), by (26.4).

In fact, we have that

$$(26.5) \quad \overline{A} \cap \overline{B} = \emptyset$$

under these conditions. To see this, we can first use the definition of a uniformity, to get $U_1, U_2 \in \mathcal{U}$ satisfying (25.12). If A, B are U -separated, then it follows that (25.15) holds as well. This implies (26.5), for the same reasons as before.

Alternatively, (26.5) is the same as saying that

$$(26.6) \quad (\overline{A} \times \overline{B}) \cap \Delta = \emptyset,$$

where Δ is as in (2.1), as usual. Of course, $\overline{A} \times \overline{B}$ is the same as the closure of $A \times B$ in $X \times X$, with respect to the product topology corresponding to the topology on X associated to \mathcal{U} . If A, B are U -separated in X and Δ is contained in the interior of U with respect to the product topology on $X \times X$, then (26.6) follows from (25.3). If $U \in \mathcal{U}$, then one can check that Δ is contained in the interior of U with respect to the product topology on $X \times X$. More precisely, the interior of U with respect to the product topology on $X \times X$ is an element of \mathcal{U} too, as in Theorem 6 on p179 of [12].

As another variant, suppose that A and B are uniformly separated in X again, and let U, U_1 , and U_2 be elements of \mathcal{U} that satisfy (25.4) and (25.12), as before. Similarly, there are $U_{1,1}, U_{1,2} \in \mathcal{U}$ that satisfy (25.17), by the definition of a uniformity. One might as well take $U_{1,1} = U_{1,2}$ here, as in (3.3). If A, B are U -separated, then (25.15) and hence (25.19) hold, so that $U_{1,1}[A]$ and $U_2[B]$ are $U_{1,2}$ -separated in X . In particular, this means that $U_{1,1}[A]$ and $U_2[B]$ are uniformly separated in X . As in Section 5 again, we have that

$$(26.7) \quad \overline{A} \subseteq U_{1,1}[A], \quad \overline{B} \subseteq U_2[B].$$

This implies that $\overline{A}, \overline{B}$ are uniformly separated in X .

Suppose that K, E are disjoint subsets of X such that K is compact and E is closed with respect to the topology on X associated to \mathcal{U} . Thus $X \setminus E$ is an open set that contains K , and hence there is a $U \in \mathcal{U}$ such that

$$(26.8) \quad U[K] \subseteq X \setminus E,$$

as in (7.1). This is the same as saying that $U[K]$ is disjoint from E , so that K, E are U -separated, as in (25.4). This shows that K and E are uniformly separated in X under these conditions.

Let Y be a subset of X , equipped with the uniform structure induced by \mathcal{U} on X as in Section 12. If A, B are subsets of Y , then one can check that A and B are uniformly separated in Y with respect to the induced uniform structure if and only if A and B are uniformly separated in X with respect to \mathcal{U} .

27 Equivalence relations

Let X be a set, and suppose for the moment that $U \subseteq X \times X$ corresponds to an equivalence relation on X . If $A \subseteq X$ and $U[A]$ is as in (2.13), then $U[A]$ is the same as the union of the equivalence classes in X associated to U that contain an element of A . Thus

$$(27.1) \quad U[U[A]] = U[A],$$

which can also be viewed in terms of (2.14) and (20.3). In particular, A is itself a union of equivalence classes in X associated to U if and only if

$$(27.2) \quad U[A] = A.$$

As in Section 25, $A, B \subseteq X$ are U -separated in X when (25.4) holds. In this case, this is the same as saying that B is disjoint from the equivalence classes in X associated to U that contain an element of A . Because U is symmetric, this is equivalent to

$$(27.3) \quad A \cap (U[B]) = \emptyset,$$

which means that A is disjoint from the equivalence classes in X associated to U that contain an element of B . This implies that

$$(27.4) \quad (U[A]) \cap (U[B]) = \emptyset,$$

which is the same as saying that the equivalence classes in X associated to U that contain elements of A and B , respectively, are disjoint. Note that (27.4) corresponds to (25.15) with $U_1 = U_2 = U$.

Let $d(\cdot, \cdot)$ be a semi-ultrametric on X , and let $U_d(r)$ be as in (2.6) for each $r > 0$. Thus $U_d(r)$ corresponds to an equivalence relation on X for every $r > 0$, as in Section 8. The equivalence classes in X associated to $U_d(r)$ are the same as open balls in X of radius r with respect to $d(\cdot, \cdot)$. By the remarks in the preceding paragraph, any two distinct equivalence classes in X with respect to $U_d(r)$ are $U_d(r)$ -separated in X . In this case, this means that any two open balls in X of radius r with respect to $d(\cdot, \cdot)$ that correspond to distinct subsets of X are r -separated in X with respect to $d(\cdot, \cdot)$.

Similarly,

$$(27.5) \quad d(x, y) \leq r$$

defines an equivalence relation on X for every $r \geq 0$ when $d(\cdot, \cdot)$ is a semi-ultrametric on X , as in Section 8. The corresponding equivalence classes in X are the closed balls in X with respect to $d(\cdot, \cdot)$ with radius r . As before, if B_1 and B_2 are distinct subsets of X that can be expressed as closed balls of radius r with respect to $d(\cdot, \cdot)$, then

$$(27.6) \quad d(x, y) > r$$

for every $x \in B_1$ and $y \in B_2$. This is the same as saying that B_1 and B_2 are separated with respect to the equivalence relation (27.5). This also works for any semimetric $d(\cdot, \cdot)$ on X when $r = 0$.

Suppose that $U \subseteq X \times X$ corresponds to an equivalence relation on X again, and let U_0 be another subset of $X \times X$ that contains the diagonal Δ . If

$$(27.7) \quad U_0 \subseteq U,$$

then any pair of distinct equivalence classes in X with respect to U are U_0 -separated in X . Conversely, if every pair of distinct equivalence classes in X with respect to U are U_0 -separated in X , then (27.7) holds. If \mathcal{U} is a uniformity on X , $U_0 \in \mathcal{U}$, and (27.7) holds, then $U \in \mathcal{U}$ too. Thus $U \in \mathcal{U}$ when distinct equivalence classes in X with respect to U are uniformly separated with respect to a single $U_0 \in \mathcal{U}$.

Let A be any subset of X , and consider

$$(27.8) \quad U_A = (A \times A) \cup ((X \setminus A) \times (X \setminus A))$$

as a subset of $X \times X$. This corresponds to the equivalence relation $x \sim_A y$ on X which is satisfied when either both x and y are elements of A or both x and y are not in A . Observe that

$$(27.9) \quad (X \times X) \setminus U_A = (A \times (X \setminus A)) \cup ((X \setminus A) \times A).$$

This means that U_A is the same as (26.3), with $B = X \setminus A$. It follows that A is uniformly separated from $X \setminus A$ with respect to a uniformity \mathcal{U} on X if and only if

$$(27.10) \quad U_A \in \mathcal{U},$$

as in the previous section.

Let Y be another set, let f be a mapping from X into Y , and let f_2 be the corresponding mapping from $X \times X$ into $Y \times Y$, as in (9.1). If $V \subseteq Y \times Y$ corresponds to an equivalence relation on Y , then

$$(27.11) \quad f_2^{-1}(V)$$

corresponds to an equivalence relation on X . In particular, if $X \subseteq Y$, then we can take $f(x) = x$ for each $x \in X$. In this case, (27.11) reduces to

$$(27.12) \quad V \cap (X \times X).$$

28 U -Chains

Let X be a set, and let U be a subset of $X \times X$. Suppose that

$$(28.1) \quad \Delta \subseteq U$$

and

$$(28.2) \quad \tilde{U} = U,$$

where Δ and \tilde{U} are as in (2.1) and (2.2), respectively. This is the same as saying that the binary relation on X corresponding to U is reflexive and symmetric.

Let n be a positive integer, and let x_1, \dots, x_n be a finite sequence of elements of X of length n . Let us say that x_1, \dots, x_n is a U -chain of length n in X if

$$(28.3) \quad (x_j, x_{j+1}) \in U$$

for each $j = 1, \dots, n-1$. This condition is considered to be vacuous when $n = 1$.

If n is a nonnegative integer, then let U^n be the set of $(x, x') \in X \times X$ for which there is a U -chain x_1, \dots, x_{n+1} of length $n+1$ such that $x_1 = x$ and $x_{n+1} = x'$. Thus $U^0 = \Delta$ and $U^1 = U$, by construction. If k, l are nonnegative integers, then it is easy to see that

$$(28.4) \quad U^k * U^l = U^{k+l},$$

where the left side of (28.4) is as defined in (2.3). Note that

$$(28.5) \quad U^n \subseteq U^{n+1}$$

for each $n \geq 0$, because of (28.1), and using either (28.4) or simply the definition of U^n . One can also check that U^n is symmetric for every $n \geq 0$, which is to say that

$$(28.6) \quad \widetilde{(U^n)} = U^n,$$

using (28.2).

Put

$$(28.7) \quad \widehat{U} = \bigcup_{n=0}^{\infty} U^n,$$

which is a subset of $X \times X$ that contains U , and hence Δ . Equivalently, \widehat{U} consists of the $(x, x') \in X \times X$ such that x can be connected to x' by a U -chain of elements of X of some finite length. Observe that

$$(28.8) \quad \widehat{U} * \widehat{U} \subseteq \widehat{U},$$

and in fact

$$(28.9) \quad \widehat{U} * \widehat{U} = \widehat{U}.$$

More precisely,

$$(28.10) \quad \widehat{U} \subseteq \widehat{U} * \widehat{U}$$

because $\Delta \subseteq \widehat{U}$. The reverse inclusion basically says that one can combine two U -chains in X to get another U -chain in X when the first U -chain ends at the point where the second U -chain begins. One can also derive (28.9) from (28.4). It is easy to see that \widehat{U} is symmetric too, so that

$$(28.11) \quad \widetilde{(\widehat{U})} = \widehat{U},$$

because of (28.6).

Equivalently, (28.8) says that the binary relation on X corresponding to \widehat{U} is transitive. It follows that this is an equivalence relation on X , since it is reflexive and symmetric as well. If the binary relation on X corresponding to U is already transitive, and hence an equivalence relation, then

$$(28.12) \quad U^n = U$$

for every $n \geq 1$, so that

$$(28.13) \quad \widehat{U} = U.$$

Let U be as before, not necessarily corresponding to a transitive relation on X . Let $x \in X$ be given, and consider the set $\widehat{U}[x]$, as in (2.16). This is the same as the set of points in X that can be connected to x by a U -chain of finite length. This can also be described as the equivalence class in X containing x determined by the equivalence relation corresponding to \widehat{U} . Let us say that X is *U -connected* if

$$(28.14) \quad \widehat{U} = X \times X,$$

which means that every pair of elements of X can be connected by a U -chain of finite length. This is the same as saying that $\widehat{U}[x] = X$ for every $x \in X$.

If $\widehat{U}[x] = X$ for any $x \in X$, then this holds for every $x \in X$, and hence X is U -connected, because \widehat{U} corresponds to an equivalence relation on X .

Similarly, let us say that $E \subseteq X$ is U -connected if every pair of elements of E can be connected by a U -chain of elements of E of finite length. This corresponds to replacing X with E in the previous discussion, and replacing U with

$$(28.15) \quad U \cap (E \times E).$$

Note that (28.15) contains the diagonal in $E \times E$ and is symmetric, because of (28.1) and (28.2).

Let $d(\cdot, \cdot)$ be a semimetric on X , or a q -semimetric on X for some $q > 0$, and let r be a positive real number. If $U = U_d(r)$ is as in (2.6), then U satisfies (28.1) and (28.2), by hypothesis. In this case, a U -chain in X may be described as an r -chain in X with respect to $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a semi-ultrametric on X , then $U_d(r)$ corresponds to an equivalence relation on X , as in Section 8.

29 Chain connectedness

Let (X, \mathcal{U}) be a uniform space, and let U be an element of \mathcal{U} that is also symmetric. Thus U satisfies (28.1) too, by definition of a uniformity. As in the previous section, X is said to be U -connected if (28.14) holds. Let us say that X is *chain connected* as a uniform space with respect to \mathcal{U} if X is U -connected for every $U \in \mathcal{U}$ such that U is symmetric. As before, this means that every pair of elements of X can be connected by a U -chain of elements of X of finite length.

Let us check that X is not chain connected if and only if there is a proper nonempty subset A of X such that A is uniformly separated from $X \setminus A$ with respect to \mathcal{U} . Suppose that A is a proper nonempty subset of X that is uniformly separated from $X \setminus A$ with respect to \mathcal{U} , so that there is a $U \in \mathcal{U}$ such that A and $X \setminus A$ are U -separated. We may as well ask that U be symmetric too, as in Section 26. It is easy to see that there is no U -chain of elements of X that connects an element of A to an element of $X \setminus A$, because A and $X \setminus A$ are U -separated in X . This implies that X is not U -connected, and hence that X is not chain connected with respect to \mathcal{U} , because $A \neq X, \emptyset$ by hypothesis.

Conversely, suppose that X is not chain connected with respect to \mathcal{U} . This means that there is a $U \in \mathcal{U}$ such that U is symmetric, and that X is not U -connected. More precisely, it follows that there are elements x, y of X that cannot be connected by a U -chain of elements of X of finite length. Let A be the set of $z \in X$ for which there is a U -chain of elements of X of finite length from x to z . Thus $x \in A$ automatically, since U satisfies (28.1) by the definition of a uniformity. We also have that $y \notin A$, by hypothesis, so that $A \neq X$. It is easy to see that A is U -separated from $X \setminus A$ under these conditions, by construction. It follows that A is uniformly separated from $X \setminus A$ with respect to \mathcal{U} , as desired.

Equivalently, X is not chain connected with respect to \mathcal{U} if there is a $U \in \mathcal{U}$ such that U is symmetric and

$$(29.1) \quad \widehat{U} \neq X \times X,$$

where \widehat{U} is as in (28.7). Remember that $U \subseteq \widehat{U}$ by construction, so that $U \in \mathcal{U}$ implies that $\widehat{U} \in \mathcal{U}$. If X is not chain connected with respect to \mathcal{U} , then it follows that \widehat{U} is an element of \mathcal{U} that corresponds to an equivalence relation on X and satisfies (29.1). Of course, if U already corresponds to an equivalence relation on X , then $\widehat{U} = U$, as in (28.13).

Thus X is not chain connected with respect to \mathcal{U} if and only if there is a $U \in \mathcal{U}$ such that U corresponds to an equivalence relation on X and

$$(29.2) \quad U \neq X \times X.$$

This means that X is chain connected with respect to \mathcal{U} if and only if the only $U \in \mathcal{U}$ that corresponds to an equivalence relation on X is $U = X \times X$.

Let us see how this reformulation of chain connectedness is related to the previous one. Let A be a subset of X , and let U_A be the corresponding subset of $X \times X$ defined in (27.8). If A is uniformly separated from its complement $X \setminus A$ in X with respect to \mathcal{U} , then U_A is an element of \mathcal{U} , as in (27.10). We have also seen that U_A automatically corresponds to an equivalence relation on X . If A is a nonempty proper subset of X , then

$$(29.3) \quad U_A \neq X \times X,$$

as in (29.2).

Conversely, suppose that there is a $U \in \mathcal{U}$ that corresponds to an equivalence relation on X and satisfies (29.2), as before. In particular, (29.2) implies that $X \neq \emptyset$, and we let A be the equivalence class associated to any element of X . Thus $A \neq \emptyset$, by construction, and $A \neq X$, because of (29.2) again. We also have that A and $X \setminus A$ are U -separated in X , as in Section 27. It follows that A and $X \setminus A$ are uniformly separated in X with respect to \mathcal{U} , because $U \in \mathcal{U}$ by hypothesis.

30 Chain connectedness, continued

Let (X, \mathcal{U}) be a uniform space again, and let E be a subset of X . Let us say that E is *chain connected* in X if for each $U \in \mathcal{U}$ such that U is symmetric, we have that E is U -connected in X . This means that every pair of elements of E can be connected by a U -chain of elements of E of finite length, as in Section 28. In particular, this reduces to the definition of chain connectedness in the previous section when $E = X$.

Let Y be a subset of X , and let \mathcal{V} be the uniformity induced on Y by \mathcal{U} on X , as in Section 12. Thus \mathcal{V} consists of the $V \subseteq Y \times Y$ that can be expressed as

$$(30.1) \quad V = U \cap (Y \times Y),$$

with $U \in \mathcal{U}$. Clearly (30.1) implies that

$$(30.2) \quad \tilde{V} = \tilde{U} \cap (Y \times Y),$$

using the notation in (2.2). If U is symmetric and V is as in (30.1), then it follows that V is symmetric too. In the other direction, if $V \in \mathcal{V}$ is symmetric, then we can choose $U \in \mathcal{U}$ so that (30.1) holds and U is symmetric, by replacing U with $U \cap \tilde{U}$ if necessary.

If $E \subseteq Y$, then it is easy to see that E is chain connected with respect to \mathcal{V} on Y if and only if E is chain connected with respect to \mathcal{U} on X . This uses the remarks in the previous paragraph, to get that symmetric elements of \mathcal{V} correspond to symmetric elements of \mathcal{U} . In particular, we can take $Y = E$, to reduce to the situation considered in the preceding section.

Equivalently, $E \subseteq X$ is not chain connected if and only if there are nonempty subsets A, B of X that are uniformly separated with respect to \mathcal{U} and satisfy

$$(30.3) \quad E = A \cup B.$$

This can be verified in essentially the same way as in the previous section, when $E = X$. Alternatively, if $E \subseteq Y \subseteq X$, then one can check that E has this property with respect to \mathcal{U} on X if and only if E has the analogous property with respect to the induced uniformity \mathcal{V} on Y . This uses the corresponding statement for uniformly separated sets mentioned in Section 26. Since the other formulation of chain connectedness has the same feature, as in the preceding paragraph, the equivalence between the two formulations can be reduced to the case where $E = X$.

If $A, B \subseteq X$ are uniformly separated with respect to \mathcal{U} , then A, B are separated subsets of X with respect to the topology associated to \mathcal{U} , as in Section 26. If $E \subseteq X$ is not chain connected with respect to \mathcal{U} , then it follows that E is not connected with respect to this topology on X . Equivalently, if E is connected with respect to this topology, then E is chain connected with respect to \mathcal{U} . It is easy to give examples of subsets of the real line that are chain connected but not connected, using the uniformity determined by the standard Euclidean metric on \mathbf{R} .

Let (Z, \mathcal{W}) be another uniform space, and let f be a uniformly continuous mapping from X into Z , as in Section 9. If A, B are uniformly separated subsets of Z , then one can check that $f^{-1}(A)$ and $f^{-1}(B)$ are uniformly separated in X . If $E \subseteq X$ is chain connected, then it follows that $f(E)$ is chain connected in Z . One can also look at this more directly in terms of the initial definition of chain connectedness.

Suppose that $E \subseteq X$ is not connected with respect to the topology on X associated to \mathcal{U} as in Section 4, so that E can be expressed as $A \cup B$, where A, B are nonempty separated subsets of X . If E is compact in X , then A, B are compact too, as in Section 24. Note that the closure \overline{B} of B in X is a closed set in X that is disjoint from A , because A and B are supposed to be separated in X . Using the compactness of A , we get that A and \overline{B} are uniformly separated in X with respect to \mathcal{U} , as in Section 26. In particular, this implies that A and

A and B are uniformly separated in X . It follows that E is not chain connected in X with respect to \mathcal{U} , because A and B are nonempty. Equivalently, if $E \subseteq X$ is compact with respect to the topology on X associated to \mathcal{U} , and if E is chain connected with respect to \mathcal{U} , then E is connected with respect to the topology on X associated to \mathcal{U} .

31 Connectedness and closure

If E is a connected subset of a topological space X , then it is well known that the closure \overline{E} of E in X is connected too. Equivalently, if \overline{E} is not connected, then E is not connected. Of course, if \overline{E} is not connected, then there are nonempty separated sets $A_1, B_1 \subseteq X$ such that

$$(31.1) \quad A_1 \cup B_1 = \overline{E}.$$

If we put

$$(31.2) \quad A = A_1 \cap E, \quad B = B_1 \cap E,$$

then it is easy to see that A, B are separated subsets of X whose union is E . The remaining point is to verify that $A, B \neq \emptyset$ under these conditions.

Let (X, \mathcal{U}) be a uniform space, and let E be a subset of X again. Also let \overline{E} be the closure of E with respect to the topology on X associated to \mathcal{U} as in Section 4. If \overline{E} is not chain connected in X with respect to \mathcal{U} , then there are nonempty uniformly separated subsets A_1, B_1 of X that satisfy (31.1). If $A, B \subseteq X$ are as in (31.2), then A, B are uniformly separated sets whose union is E . As before, one can check that $A, B \neq \emptyset$, to get that E is not chain connected with respect to \mathcal{U} .

Equivalently, if E is chain connected with respect to \mathcal{U} , then \overline{E} is chain connected with respect to \mathcal{U} . Alternatively, let U be an arbitrary symmetric element of \mathcal{U} . If x is any element of \overline{E} , then there is a $w \in E$ such that

$$(31.3) \quad (x, w) \in U,$$

because of (4.13). If E is U -connected, as in Section 28, then one can use this to verify that \overline{E} is U -connected too. If E is chain connected in X , then it follows that \overline{E} is chain connected in X as well.

Suppose now that E is not chain connected in X , so that there are nonempty uniformly separated sets $A, B \subseteq X$ whose union is equal to E . This implies that the closures $\overline{A}, \overline{B}$ of A, B are uniformly separated in X , as in Section 26. Of course, $\overline{A}, \overline{B} \neq \emptyset$, because $A, B \neq \emptyset$, and it is well known that

$$(31.4) \quad \overline{E} = \overline{A} \cup \overline{B}$$

when $E = A \cup B$. It follows that \overline{E} is not chain connected in X when E is not chain connected in X , which is the same as saying that E is chain connected in X when \overline{E} is chain connected in X . Note that the analogous statement for ordinary connectedness in topological spaces does not hold.

32 Two related uniformities

Let (X, \mathcal{U}) be a uniform space, and put

$$(32.1) \quad \mathcal{B}_{eq} = \{U \in \mathcal{U} : U \text{ corresponds to an equivalence relation on } X\}.$$

Note that

$$(32.2) \quad X \times X \in \mathcal{B}_{eq},$$

because $X \times X \in \mathcal{U}$ by the definition of a uniformity, and $X \times X$ corresponds trivially to an equivalence relation on X . In particular, $\mathcal{B}_{eq} \neq \emptyset$. If $U_1, U_2 \in \mathcal{B}_{eq}$, then it is easy to see that

$$(32.3) \quad U_1 \cap U_2 \in \mathcal{B}_{eq},$$

using the corresponding property (3.4) for \mathcal{U} , and the analogous statement for equivalence relations, mentioned in Section 20. It follows that \mathcal{B}_{eq} is a base for a uniformity

$$(32.4) \quad \mathcal{U}_{eq}$$

on X , as in Section 20 again.

Similarly, put

$$(32.5) \quad \mathcal{B}_{eq,2} = \{U \in \mathcal{B}_{eq} : \text{there are only finitely many equivalence classes in } X \text{ corresponding to } U\},$$

which is to say that there are only finitely many equivalence classes in X determined by the equivalence relation on X corresponding to each $U \in \mathcal{B}_{eq,2}$. As before,

$$(32.6) \quad X \times X \in \mathcal{B}_{eq,2},$$

because X itself is the only equivalence class determined by the trivial equivalence relation corresponding to $X \times X$. If $U_1, U_2 \in \mathcal{B}_{eq,2}$, then one can check that

$$(32.7) \quad U_1 \cap U_2 \in \mathcal{B}_{eq,2},$$

as in (32.3). More precisely, the number of equivalence classes in X corresponding to $U_1 \cap U_2$ is less than or equal to the product of the numbers of equivalence classes in X corresponding to U_1 and U_2 , respectively. It follows again that $\mathcal{B}_{eq,2}$ is a base for a uniformity

$$(32.8) \quad \mathcal{U}_{eq,2}$$

on X , as in Section 20.

By construction,

$$(32.9) \quad \mathcal{B}_{eq,2} \subseteq \mathcal{B}_{eq} \subseteq \mathcal{U},$$

and hence

$$(32.10) \quad \mathcal{U}_{eq,2} \subseteq \mathcal{U}_{eq} \subseteq \mathcal{U}.$$

In particular, this implies that the topology on X associated to $\mathcal{U}_{eq,2}$ as in Section 4 is contained in the topology on X associated to \mathcal{U}_{eq} , which is itself contained in the topology on X associated to \mathcal{U} . The topology on X associated

to $\mathcal{U}_{eq,2}$ is actually the same as the topology on X associated to \mathcal{U}_{eq} , as we shall see in a moment.

Let U_A be as in (27.8) for each $A \subseteq X$, which corresponds to an equivalence relation on X . If A is a proper nonempty subset of X , then there are exactly two equivalence classes in X corresponding to U_A , which are A and $X \setminus A$. Otherwise, if $A = \emptyset$ or $A = X$, then $U_A = X \times X$. Remember that A is uniformly separated from $X \setminus A$ with respect to \mathcal{U} if and only if $U_A \in \mathcal{U}$, as in (27.10). In this case, we get that

$$(32.11) \quad U_A \in \mathcal{B}_{eq,2},$$

since there are at most two equivalence classes in X corresponding to U_A .

In particular, if $A \subseteq X$ is uniformly separated from $X \setminus A$ with respect to \mathcal{U} , then A is uniformly separated from $X \setminus A$ with respect to $\mathcal{U}_{eq,2}$, by (32.11). Of course, if A is uniformly separated from $X \setminus A$ with respect to $\mathcal{U}_{eq,2}$, then A is uniformly separated from $X \setminus A$ with respect to \mathcal{U}_{eq} , because of the first inclusion in (32.10). Similarly, if A is uniformly separated with respect to $X \setminus A$ with respect to \mathcal{U}_{eq} , then A is uniformly separated from $X \setminus A$ with respect to \mathcal{U} , by the second inclusion in (32.10). Thus \mathcal{U} , \mathcal{U}_{eq} , and $\mathcal{U}_{eq,2}$ determine the same collection of subsets A of X that are uniformly separated from their complements in X .

Suppose that $A \subseteq X$ is an equivalence class corresponding to an element V of \mathcal{B}_{eq} . This implies that A is uniformly separated from $X \setminus A$ with respect to \mathcal{U} , so that (32.11) holds. Using this, one can check that the topology on X associated to \mathcal{U}_{eq} is the same as the topology on X associated to $\mathcal{U}_{eq,2}$.

Of course, (32.11) says that

$$(32.12) \quad \{U_A : A \subseteq X \text{ is uniformly separated from } X \setminus A \text{ with respect to } \mathcal{U}\}$$

is contained in $\mathcal{B}_{eq,2}$. If V is any element of $\mathcal{B}_{eq,2}$ and $A \subseteq X$ is an equivalence class associated to V , then U_A is an element of (32.12), as in the previous paragraph. In this case, there are only finitely many such equivalence classes, by the definition (32.5) of $\mathcal{B}_{eq,2}$. This implies that V can be expressed as the intersection of finitely many elements of (32.12). It follows that (32.12) is a sub-base for $\mathcal{U}_{eq,2}$.

Observe that

$$(32.13) \quad X \text{ is totally bounded with respect to } \mathcal{U}_{eq,2}$$

automatically. More precisely, in order to verify that X is totally bounded with respect to $\mathcal{U}_{eq,2}$, it suffices to check that X satisfies the condition in Section 11 for each $U \in \mathcal{B}_{eq,2}$, because $\mathcal{B}_{eq,2}$ is a base for $\mathcal{U}_{eq,2}$. In this case, this condition follows from the requirement that there be only finitely many equivalence classes in X corresponding to U . If X is totally bounded with respect to \mathcal{U}_{eq} , then it is easy to see that there can only be finitely many equivalence classes in X corresponding to any $U \in \mathcal{B}_{eq}$. This implies that

$$(32.14) \quad \mathcal{U}_{eq} = \mathcal{U}_{eq,2},$$

which holds in particular when X is totally bounded with respect to \mathcal{U} .

33 Topological dimension 0

A topological space X is said to have *topological dimension 0* at a point $x \in X$ if for every open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$, $U \subseteq W$, and U is also a closed set in X . Equivalently, this means that there is a local base for the topology of X at x consisting of subsets of X that are both open and closed.

If X has topological dimension 0 at every point $x \in X$, then one simply says that X has topological dimension 0, at least in a strict sense. This is the same as saying that there is a base for the topology of X consisting of subsets of X that are both open and closed. Sometimes one also requires X to be nonempty to have topological dimension 0, in order to define related conditions in other dimensions inductively, but we shall not pursue this here. If X has topological dimension 0 in this strict sense, then X is regular in the strict sense, as in Section 5. In particular, if X satisfies the first or 0th separation condition as well, then X is Hausdorff, as before.

Suppose now that (X, \mathcal{U}) is a uniform space, so that X is equipped with the topology associated to \mathcal{U} as in Section 4 too. If $A \subseteq X$ is uniformly separated from $X \setminus A$ with respect to \mathcal{U} , then A and $X \setminus A$ are separated in X with respect to this topology on X , as in Section 26, which means that A is both open and closed in X . Let us say that X is *strongly 0-dimensional* at a point $x \in X$ if for each open set $W \subseteq X$ with $x \in W$ there is a set $U \subseteq X$ such that $x \in U$, $U \subseteq W$, and U is uniformly separated from $X \setminus U$ in X with respect to \mathcal{U} . This implies that X has topological dimension 0 at x , since U is both open and closed in X under these conditions. Equivalently, X is strongly 0-dimensional at x if there is a local base for the topology of X at x consisting of subsets of X that are uniformly separated from their complements in X with respect to \mathcal{U} .

If X is strongly 0-dimensional at every point $x \in X$, then let us say that X is strongly 0-dimensional as a uniform space. This implies that X has topological dimension 0, as before. Equivalently, X is strongly 0-dimensional if there is a base for the topology of X consisting of subsets of X that are uniformly separated from their complements in X with respect to \mathcal{U} .

Let us say that X is *uniformly 0-dimensional* if there is a base \mathcal{B} for \mathcal{U} such that each $U \in \mathcal{B}$ corresponds to an equivalence relation on X . It suffices to have a sub-base for \mathcal{U} with this property, since the intersection of two subsets of $X \times X$ corresponding to equivalence relations on X corresponds to an equivalence relation on X as well. If X is uniformly 0-dimensional with respect to \mathcal{U} , then one can check that X is strongly 0-dimensional with respect to \mathcal{U} . This uses the fact that if $U \in \mathcal{U}$ corresponds to an equivalence relation on X , then each equivalence class in X determined by U is U -separated from its complement. Note that X is uniformly 0-dimensional with respect to \mathcal{U} if and only if \mathcal{U} corresponds to a nonempty collection of semi-ultrametrics on X as in Section 18, by the discussion in Section 20.

If \mathcal{U}_{eq} and $\mathcal{U}_{eq,2}$ are the uniformities obtained from \mathcal{U} as in the previous section, then X is automatically uniformly 0-dimensional with respect to both

\mathcal{U}_{eq} and $\mathcal{U}_{eq,2}$. Similarly,

$$(33.1) \quad \mathcal{U} = \mathcal{U}_{eq}$$

if and only if X is uniformly 0-dimensional with respect to \mathcal{U} . In particular, X is automatically strongly 0-dimensional with respect to both \mathcal{U}_{eq} and $\mathcal{U}_{eq,2}$, as in the preceding paragraph. Remember that the topologies on X associated to \mathcal{U}_{eq} and $\mathcal{U}_{eq,2}$ are the same, as in the previous section. The condition that X be strongly 0-dimensional with respect to \mathcal{U} is equivalent to saying that the topology on X associated to \mathcal{U} is the same as the topology on X associated to \mathcal{U}_{eq} or $\mathcal{U}_{eq,2}$.

A topological space X is said to be *totally separated* if for every $x, y \in X$ with $x \neq y$ there is an open set $U \subseteq X$ such that $x \in U$, $y \notin U$, and U is a closed set in X too. In particular, this implies that X is Hausdorff. If X has topological dimension 0 and satisfies the first or 0th separation condition, then it is easy to see that X is totally separated.

Let us say that a uniform space (X, \mathcal{U}) is *strongly totally separated* if for each $x, y \in X$ with $x \neq y$ there is a set $U \subseteq X$ such that $x \in U$, $y \notin U$, and U is uniformly separated from $X \setminus U$ in X with respect to \mathcal{U} . This implies that U is both open and closed in X with respect to the topology associated to \mathcal{U} , as before. If X is strongly totally separated with respect to \mathcal{U} , then it follows that X is totally separated with respect to the topology associated to \mathcal{U} . If X is strongly 0-dimensional with respect to \mathcal{U} , and if X satisfies the first or 0th separation condition with respect to the topology associated to \mathcal{U} , then X is strongly totally separated with respect to \mathcal{U} . Note that X is strongly totally separated with respect to \mathcal{U} if and only if X is Hausdorff with respect to the topology associated to \mathcal{U}_{eq} or $\mathcal{U}_{eq,2}$.

As a basic example, the set \mathbf{Q} of rational numbers has topological dimension 0 with respect to the standard Euclidean topology. However, \mathbf{Q} is also chain connected with respect to the uniformity determined by the standard Euclidean metric. This implies that there are no nonempty proper subsets of \mathbf{Q} that are uniformly separated from their complements in \mathbf{Q} with respect to this uniformity. In particular, this means that \mathbf{Q} is not strongly 0-dimensional at any point with respect to this uniformity. Similarly, \mathbf{Q} is not strongly totally separated with respect to this uniformity.

Part III

Topological groups

34 Basic notions

Let G be a group, with the group operations expressed multiplicatively. Suppose that G is also equipped with a topology, so that $G \times G$ may be equipped with the corresponding product topology. If the group operations on G are continuous,

then G is said to be a *topological group*. More precisely, this means that

$$(34.1) \quad (x, y) \mapsto x y$$

should be continuous as a mapping from $G \times G$ into G . Similarly, the mapping

$$(34.2) \quad x \mapsto x^{-1}$$

should be continuous on G as well. This implies that (34.2) should be a homeomorphism on G , since this mapping is its own inverse. Sometimes one also asks that $\{e\}$ be a closed set in G , where e is the identity element in G .

Put

$$(34.3) \quad L_a(x) = a x$$

and

$$(34.4) \quad R_a(x) = x a$$

for every $a, x \in G$. These define the *left* and *right translation* mappings on G associated to $a \in G$. If G is a topological group, then L_a and R_a define continuous mappings on G for each $a \in G$, which corresponds to continuity of multiplication on G in each variable separately. Note that L_a and R_a are one-to-one mappings from G onto itself for each $a \in G$, with inverse mappings $L_{a^{-1}}$ and $R_{a^{-1}}$, respectively. If G is a topological group, then it follows that L_a and R_a are homeomorphisms on G for every $a \in G$. Of course, if G is commutative, then L_a is the same as R_a for every $a \in G$. If $\{e\}$ is a closed set in G , then continuity of left or right translations implies that G satisfies the first separation condition.

If $a \in G$ and $E \subseteq G$, then we put

$$(34.5) \quad a E = L_a(E) \quad \text{and} \quad E a = R_a(E).$$

Also put

$$(34.6) \quad E^{-1} = \{x^{-1} : x \in E\},$$

which is the image of E under (34.2). If $A, B \subseteq G$, then we put

$$(34.7) \quad A B = \{a b : a \in A, b \in B\}.$$

Equivalently,

$$(34.8) \quad A B = \bigcup_{a \in A} a B = \bigcup_{b \in B} A b.$$

If G is a topological group and either A or B is an open set, then it follows that $A B$ is an open set too, by continuity of translations, and because a union of open sets is an open set.

Continuity of (34.1) as a mapping from $G \times G$ into G at e means that for each open set $W \subseteq G$ with $e \in W$, there are open sets $U, V \subseteq G$ that both contain e and satisfy

$$(34.9) \quad U V \subseteq W.$$

Suppose that G is equipped with a topology such that the left and right translation mappings L_a and R_a are continuous for every $a \in G$, and hence are homeomorphisms. If (34.1) is continuous as a mapping from $G \times G$ into G at (e, e) , then it is easy to see that (34.1) is continuous everywhere on $G \times G$. Similarly, if (34.2) is continuous at e , then (34.2) is continuous everywhere on G under these conditions.

35 Associated uniformities

Let G be a group, and let A be a subset of G . Put

$$(35.1) \quad \begin{aligned} A_L &= \{(x, y) \in G \times G : x^{-1}y \in A\} \\ &= \{(x, y) \in G \times G : y = xa \text{ for some } a \in A\} \end{aligned}$$

and

$$(35.2) \quad \begin{aligned} A_R &= \{(x, y) \in G \times G : yx^{-1} \in A\} \\ &= \{(x, y) \in G \times G : y = ax \text{ for some } a \in A\}. \end{aligned}$$

Of course, $A_L = A_R$ when G is commutative. If $\Delta = \Delta_G$ is the diagonal in $G \times G$, as in (2.1), then

$$(35.3) \quad \Delta \subseteq A_L, A_R$$

exactly when $e \in A$. We also have that

$$(35.4) \quad \widetilde{(A_L)} = (A^{-1})_L \quad \text{and} \quad \widetilde{(A_R)} = (A^{-1})_R,$$

where the left sides of these equations are defined as in (2.2). By construction,

$$(35.5) \quad A_L[x] = xA \quad \text{and} \quad A_R[x] = Ax$$

for every $x \in G$, using the notation in (2.16). Similarly,

$$(35.6) \quad A_L[E] = EA \quad \text{and} \quad A_R[E] = AE$$

for every $E \subseteq G$, using the notation in (2.13). If B is another subset of G , then

$$(35.7) \quad A_L * B_L = (AB)_L \quad \text{and} \quad A_R * B_R = (BA)_R,$$

where the left sides of these equations are as defined in (2.3).

Suppose now that G is a topological group. One can check that

$$(35.8) \quad \mathcal{B}_L = \{W_L : W \subseteq G \text{ is an open set with } e \in W\}$$

is a base for a uniformity on G , which is known as the *left uniformity* on G . Similarly,

$$(35.9) \quad \mathcal{B}_R = \{W_R : W \subseteq G \text{ is an open set with } e \in W\}$$

is a base for a uniformity on G , which is known as the *right uniformity* on G . Note that W_R is defined a bit differently on p210 of [12], and corresponds to $(W^{-1})_R$ here. This does not affect (35.9), because W is an open subset of G that contains e if and only if W^{-1} has the same properties.

It is easy to see that the given topology on G is the same as the topology associated to the corresponding left or right uniformity on G . In particular, this implies that G is regular as a topological space in the strict sense, as in Section 5. If $\{e\}$ is a closed set in G , then G satisfies the first separation condition, as in the previous section, and hence G is Hausdorff as a topological space.

Suppose that G_1 and G_2 are topological groups, and that ϕ is a continuous group homomorphism from G_1 into G_2 . Under these conditions, it is easy to see that ϕ is uniformly continuous as a mapping from G_1 into G_2 , with respect to their corresponding left uniformities. Similarly, ϕ is uniformly continuous with respect to the corresponding right uniformities on G_1 and G_2 . More precisely, if ϕ is continuous at the identity element in G_1 , then ϕ is continuous, and uniformly continuous with respect to the left and right uniformities on G_1 and G_2 , respectively.

36 Some additional properties

Let G be a group again, and let L_g, R_g be the left and right translation mappings on G corresponding to $g \in G$ as in (34.3) and (34.4). This leads to mappings $L_{g,2}$ and $R_{g,2}$ from $G \times G$ into itself as in (9.1), so that

$$(36.1) \quad L_{g,2}(x, y) = (L_g(x), L_g(y)) = (g x, g y)$$

and

$$(36.2) \quad R_{g,2}(x, y) = (R_g(x), R_g(y)) = (x g, y g)$$

for every $g, x, y \in G$. If $A \subseteq G$ and $A_L, A_R \subseteq G \times G$ are as in (35.1), (35.2), then it is easy to see that

$$(36.3) \quad L_{g,2}(A_L) = A_L \quad \text{and} \quad R_{g,2}(A_R) = A_R$$

for every $g \in G$. In particular, if G is a topological group, then each element of \mathcal{B}_L in (35.8) is invariant under $L_{g,2}$ for every $g \in G$, and each element of \mathcal{B}_R in (35.9) is invariant under $R_{g,2}$ for every $g \in G$.

Similarly, one can check that

$$(36.4) \quad R_{g,2}(A_L) = (g^{-1} A g)_L \quad \text{and} \quad L_{g,2}(A_R) = (g A g^{-1})_R$$

for every $g \in G$ and $A \subseteq G$. If G is a topological group, then it follows that R_g is uniformly continuous with respect to the left uniformity on G for every $g \in G$, and that L_g is uniformly continuous with respect to the right uniformity on G for every $g \in G$.

Let j denote the mapping (34.2) that sends an element of G to its inverse, so that

$$(36.5) \quad j(A) = A^{-1}$$

for each $A \subseteq G$, by construction. Also let j_2 be the corresponding mapping from $G \times G$ into itself, as in (9.1), so that

$$(36.6) \quad j_2(x, y) = (x^{-1}, y^{-1})$$

for every $x, y \in G$. Observe that

$$(36.7) \quad j_2(A_L) = (A^{-1})_R \quad \text{and} \quad j_2(A_R) = (A^{-1})_L$$

for every $A \subseteq G$. In particular, if G is a topological group, then j_2 sends elements of \mathcal{B}_L to elements of \mathcal{B}_R , and vice-versa. This implies that j is uniformly continuous as a mapping from G with the left uniformity to G with the right uniformity, and vice-versa.

If $g \in G$, then

$$(36.8) \quad C_g(x) = g x g^{-1}$$

defines an (inner) automorphism on G , which is conjugation by g . Of course,

$$(36.9) \quad C_g(A) = g A g^{-1}$$

for each $A \subseteq G$. Let $C_{g,2}$ be the mapping from $G \times G$ into itself corresponding to C_g as in (9.1), so that

$$(36.10) \quad C_{g,2}(x, y) = (C_g(x), C_g(y)) = (g x g^{-1}, g y g^{-1})$$

for every $x, y \in G$. Observe that

$$(36.11) \quad C_{g,2}(A_L) = (C_g(A))_L \quad \text{and} \quad C_{g,2}(A_R) = (C_g(A))_R$$

for every $A \subseteq G$, which can also be derived from (36.3) and (36.4). If G is a topological group, then it follows that C_g is uniformly continuous with respect to the left and right uniformities on G , which could be obtained from the earlier statements for left and right translations as well.

37 Translation-invariant relations

Let G be a group, and let $U \subseteq G \times G$ be given. Let us say that U is invariant under left translations on G if

$$(37.1) \quad L_{g,2}(U) = U$$

for every $g \in G$, where $L_{g,2}$ is as in (36.1). Similarly, U is invariant under right translations on G if

$$(37.2) \quad R_{g,2}(U) = U$$

for every $g \in G$, where $R_{g,2}$ is as in (36.2). Put

$$(37.3) \quad A = U[e],$$

where the right side of (37.3) is defined as in (2.16). If U is invariant under left translations on G , then it is easy to see that

$$(37.4) \quad U = A_L,$$

where A_L is as in (35.1). In the same way, if U is invariant under right translations on G , then

$$(37.5) \quad U = A_R,$$

where A_R is as in (35.2). Of course, for any $A \subseteq G$, we have seen that A_L and A_R are invariant under left and right translations on G , respectively, as in (36.1) and (36.2).

Let us say that $A \subseteq G$ is invariant under conjugation if

$$(37.6) \quad C_g(A) = A$$

for every $g \in G$, where C_g is as in (36.8). Similarly, we say that $U \subseteq G \times G$ is invariant under conjugations on G when

$$(37.7) \quad C_{g,2}(U) = U$$

for every $g \in G$, where $C_{g,2}$ is as in (36.10). If $A \subseteq G$ is invariant under conjugation in the first sense, then A_L and A_R are invariant under conjugation in the second sense, by (36.11). This is basically the same as saying that A_L and A_R are invariant under right and left translations, respectively, as in (36.4). In this case, we also have that

$$(37.8) \quad A_L = A_R.$$

If $U \subseteq G \times G$ is invariant under conjugations and A is as in (37.3), then A is invariant under conjugation as a subset of G . If U is invariant under both left and right translations, then U is invariant under conjugations in particular. If U is invariant under either left or right translations, and if U is invariant under conjugations, then U is invariant under both left and right translations.

If A is a subgroup of G , then A_L and A_R correspond to equivalence relations on G . The equivalence classes associated to these equivalence relations are the usual left and right cosets of A in G , respectively. Remember that A is said to be normal in G exactly when A is invariant under conjugation, in which case the left and right cosets of A in G are the same. This implies that (37.8) holds, and the corresponding equivalence relation on G is invariant under both left and right translations, as in the previous paragraph.

Suppose now that $U \subseteq G \times G$ is invariant under left or right translations, so that U can be given as in (37.4) or (37.5), respectively, with A as in (37.3). In both cases, if U also corresponds to an equivalence relation on G , then one can check that A is a subgroup of G . If U is invariant under both left and right translations, and hence under conjugations, then A is a normal subgroup of G .

38 Translation-invariant semimetrics

Let G be a group, and let $d(x, y)$ be a q -semimetric on G for some $q > 0$. We say that $d(\cdot, \cdot)$ is invariant under left translations on G if

$$(38.1) \quad d(ax, ay) = d(x, y)$$

for every $a, x, y \in G$. Similarly, $d(\cdot, \cdot)$ is invariant under right translations on G if

$$(38.2) \quad d(xa, ya) = d(x, y)$$

for every $a, x, y \in G$. In both cases, we get that

$$(38.3) \quad d(x, e) = d(x^{-1}, e)$$

for every $x \in G$, using the symmetry condition (1.2). Note that $d(x, y)$ is invariant under left translations on G if and only if

$$(38.4) \quad d(x^{-1}, y^{-1})$$

is invariant under right translations on G .

If

$$(38.5) \quad d(axa^{-1}, aya^{-1}) = d(x, y)$$

for every $a, x, y \in G$, then $d(\cdot, \cdot)$ is said to be invariant under conjugations on G . In particular, this implies that

$$(38.6) \quad d(axa^{-1}, e) = d(x, e)$$

for every $a, x \in G$. If $d(\cdot, \cdot)$ is invariant under left or right translations on G and satisfies (38.6), then one can check that $d(\cdot, \cdot)$ is invariant under conjugations, and in fact under both left and right translations on G . Of course, if $d(\cdot, \cdot)$ is invariant under both left and right translations on G , then $d(\cdot, \cdot)$ is invariant under conjugations on G . In this case, one can verify that

$$(38.7) \quad d(x^{-1}, y^{-1}) = d(x, y)$$

for every $x, y \in G$ too.

Let $r > 0$ be given, and let $U_d(r) \subseteq G \times G$ be as in (2.6). If $d(\cdot, \cdot)$ is invariant under left or right translations on G , then $U_d(r)$ has the analogous property, as defined in the preceding section. Similarly, if $d(\cdot, \cdot)$ is invariant under conjugations on G , then $U_d(r)$ is invariant under conjugations as well. This implies that the open ball $B_d(e, r)$ defined as in (1.5) is invariant under conjugations as a subset of G , since

$$(38.8) \quad B_d(e, r) = (U_d(r))[e],$$

as in (37.3). More precisely, $B_d(e, r)$ is invariant under conjugations when (38.6) holds. If $d(\cdot, \cdot)$ is a semi-ultrametric on G , then $U_d(r)$ corresponds to an equivalence relation on G , as in Section 8. If $d(\cdot, \cdot)$ is a semi-ultrametric on G that is

invariant under left or right translations on G , then it follows that $B_d(e, r)$ is a subgroup of G , as in the previous section. If $d(\cdot, \cdot)$ is a semi-ultrametric on G that is invariant under both left and right translations on G , then $B_d(e, r)$ is a normal subgroup of G .

If A is a subgroup of G , then A_L and A_R in (35.1) and (35.2) correspond to equivalence relations on G , as in the previous section. These equivalence relations are also invariant under left and right translations, respectively. It is easy to see that the corresponding discrete semi-ultrametrics on G as in (8.8) are invariant under left and right translations as well. If A is a normal subgroup of G , then $A_L = A_R$, and the corresponding discrete semi-ultrametric is invariant under both left and right translations on G .

39 Translation-invariance and topology

Suppose now that G is a topological group. Let A be an open subset of G that contains e , and let A_L, A_R be as in (35.1), (35.2), respectively. Thus A_L, A_R are elements of $\mathcal{B}_L, \mathcal{B}_R$ in (35.8), (35.9), respectively. In particular, A_L and A_R are elements of the corresponding left and right uniformities on G , respectively. If A is a subgroup of G , then A_L, A_R correspond to equivalence relations on G , as before. Note that a subgroup A of G is an open subset of G when e is an element of the interior of A , by continuity of translations.

Conversely, if $U \subseteq G \times G$ is invariant under left or right translations, then we have seen that U can be expressed as A_L or A_R , respectively, where A is as in (37.3). If U corresponds to an equivalence relation on G , then A is a subgroup of G . If U is an element of the left or right uniformity on G , then it is easy to see that e is an element of the interior of (37.3) in G . This uses the way that the left and right uniformities on G are defined in terms of the given topology on G . If U has each of these three properties, then it follows that A is an open subgroup of G .

Let \mathcal{A} be a collection of open subsets of G that contain e as an element. Put

$$(39.1) \quad \mathcal{A}_L = \{A_L : A \in \mathcal{A}\}$$

and

$$(39.2) \quad \mathcal{A}_R = \{A_R : A \in \mathcal{A}\},$$

so that $\mathcal{A}_L \subseteq \mathcal{B}_L$ and $\mathcal{A}_R \subseteq \mathcal{B}_R$, by definition of \mathcal{B}_L and \mathcal{B}_R . If \mathcal{A} is the collection of all open subsets of G that contain e , then $\mathcal{A}_L = \mathcal{B}_L$ and $\mathcal{A}_R = \mathcal{B}_R$. Similarly, if \mathcal{A} is a local base for the topology of G at e , then \mathcal{A}_L and \mathcal{A}_R form bases for the left and right uniformities on G , respectively. Conversely, if \mathcal{A}_L or \mathcal{A}_R is a base for the left or right uniformity on G , respectively, then it is easy to see that \mathcal{A} has to be a local base for the topology of G at e .

If \mathcal{A} is a local base for the topology of e consisting of open subgroups of G , then \mathcal{A}_L and \mathcal{A}_R are bases for the left and right uniformities on G , respectively, whose elements correspond to equivalence relations on G . This implies that G is uniformly 0-dimensional with respect to the left and right uniformities, as in Section 33. The converse will be discussed in the next section.

It is well known that there is a collection of semimetrics on G that are invariant under left translations on G , and for which the corresponding topology on G is the given topology. This implies that the uniformity on G determined by this collection of semimetrics is the same as the left uniformity on G , because of invariance under left translations. If there is a local base for the topology of G at e with only finitely or countably many elements, then there is a single semimetric on G that is invariant under left translations and determines the same topology on G , and for which the corresponding uniformity is hence the left uniformity. Of course, there are analogous statements for right translations and the right uniformity.

Suppose that the topology on G is determined by a nonempty collection \mathcal{M} of semi-ultrametrics on G that are invariant under left or right translations. Of course, the open balls in G centered at e with respect to elements of \mathcal{M} are open subsets of G under these conditions, and they are also subgroups of G , as in the previous section. This implies that the open subgroups of G form a local base for the topology of G at e . If the elements of \mathcal{M} are invariant under both left and right translations, then the corresponding open balls in G centered at e are normal subgroups of G . In this case, the open normal subgroups of G form a local base for the topology of G at e .

Conversely, suppose that \mathcal{A} is a collection of open subgroups of G that is a local base for the topology of G at e . If $A \in \mathcal{A}$, then we get subsets A_L, A_R of $G \times G$ corresponding to equivalence relations on G as before. The discrete semi-ultrametrics on G associated to these equivalence relations are invariant under left and right translations, respectively, as in the previous section. By doing this for each $A \in \mathcal{A}$, we get collections of semi-ultrametrics on G that are invariant under left or right translations and which determine the same topology on G . If the elements of \mathcal{A} are normal subgroups of G , then the corresponding discrete semi-ultrametrics on G are invariant under both left and right translations, as before.

If d_1, d_2, \dots, d_n are finitely many semimetrics on G , each of which is invariant under left translations, then their sum and maximum are invariant under left translations too. If d is a semimetric on G that is invariant under left translations and t is a positive real number, then the semimetric d_t defined on G as in (19.1) is invariant under left translations as well. Similarly, if d_1, d_2, d_3, \dots is an infinite sequence of semimetrics on G , each of which is invariant under left translations, then the semimetric d' defined on G as in (19.6) is also invariant under left translations. There are analogous statements for right translations, as usual.

40 Translation-invariant relations, continued

Let G be a group, and let U be a subset of $G \times G$. Observe that

$$(40.1) \quad \bigcap_{g \in G} L_{g,2}(U) = \{(x, y) \in G \times G : (g^{-1}x, g^{-1}y) \in U \text{ for every } g \in G\}$$

where $L_{g,2}$ is as in (36.1) for each $g \in G$. Equivalently,

$$(40.2) \quad \bigcap_{g \in G} L_{g,2}(U) = \{(x, y) \in G \times G : (h^{-1}, h^{-1} x^{-1} y) \in U \text{ for every } h \in G\},$$

where h corresponds to $x^{-1}g$ in the right side of (40.1). This implies that

$$(40.3) \quad \bigcap_{g \in G} L_{g,2}(U) = \left(\bigcap_{h \in G} h U [h^{-1}] \right)_L,$$

where the right side of (40.3) is defined as in (35.1), as in Section 37. Of course, the left side of (40.3) is invariant under left translations on G as a subset of $G \times G$ by construction. One can also check directly that

$$(40.4) \quad \left(\bigcap_{g \in G} L_{g,2}(U) \right) [e] = \bigcap_{g \in G} g U [g^{-1}],$$

where the definition (2.16) is used on both sides of the equation. Thus (40.3) is the same as (37.4) in this situation.

Suppose now that G is a topological group, and that U is an element of the corresponding left uniformity on G , as in Section 35. This means that there is an open set $W \subseteq G$ such that $e \in W$ and

$$(40.5) \quad W_L \subseteq U,$$

where W_L is as in (35.1) again. It follows that

$$(40.6) \quad W_L \subseteq \bigcap_{g \in G} L_{g,2}(U),$$

because W_L is automatically invariant under left translations, as in (36.3). This is basically the same as saying that

$$(40.7) \quad W \subseteq \bigcap_{g \in G} g U [g^{-1}],$$

because of (40.3). Note that (40.6) implies that (40.1) is an element of the left uniformity on G too.

If U corresponds to an equivalence relation on G , then it is easy to see that (40.1) corresponds to an equivalence relation on G as well. This implies that (40.4) is a subgroup of G , as in Section 37, because (40.1) is invariant under left translations. If G is a topological group, and if U is an element of the left uniformity, then e is an element of the interior of this subgroup, by (40.7). It follows that (40.4) is an open subgroup of G under these conditions. Note that (40.1) is automatically contained in U .

To summarize, if $U \subseteq G \times G$ is an element of the left uniformity on G , and if U corresponds to an equivalence relation on G , then there is an open subgroup A of L such that

$$(40.8) \quad A_L \subseteq U.$$

More precisely, one can take A to be (40.4), so that A_L is as in (40.3). If \mathcal{A} is the collection of open subgroups of G and \mathcal{A}_L is as in (39.1), then it follows that \mathcal{A}_L is a base for the uniformity on G associated to the left uniformity on G as in (32.4). In particular, if G is uniformly 0-dimensional with respect to the left uniformity, as in Section 33, then it follows that there is a local base for the topology of G at e consisting of open subgroups of G . Of course, one can deal with right translations and the right uniformity on G in the same way.

41 Uniform separation

Let G be a group, let W be a subset of G that contains e , and let W_L, W_R be defined as in (35.1), (35.2), as usual. Note that W_L and W_R contain the diagonal in $G \times G$ under these conditions, as in (35.3). Remember that

$$(41.1) \quad W_L[E] = EW \quad \text{and} \quad W_R[E] = WE$$

for every $E \subseteq G$, as in (35.6), and using the notation in (2.13). It follows that $E_1, E_2 \subseteq G$ are W_L -separated in the terminology of Section 25 exactly when

$$(41.2) \quad (E_1 W) \cap E_2 = \emptyset,$$

as in (25.4). Similarly, E_1, E_2 are W_R -separated exactly when

$$(41.3) \quad (W E_1) \cap E_2 = \emptyset.$$

Suppose now that G is a topological group, so that uniform separation of subsets of G with respect to the left and right uniformities can be defined as in Section 26. More precisely, $E_1, E_2 \subseteq G$ are uniformly separated with respect to the left uniformity exactly when (41.2) holds for some open set $W \subseteq G$ that contains e . Similarly, E_1, E_2 are uniformly separated with respect to the right uniformity on G exactly when (41.3) holds for some open set $W \subseteq G$ that contains e .

Let E be a subset of G , and let us apply the previous remarks to E and its complement in G . It follows that E is uniformly separated from its complement with respect to the left uniformity on G if and only if

$$(41.4) \quad EW \subseteq E$$

for some open set $W \subseteq G$ that contains e . Similarly, E is uniformly separated from its complement with respect to the right uniformity on G if and only if

$$(41.5) \quad WE \subseteq E$$

for some open set $W \subseteq G$ that contains e . In both cases, the inclusion is actually an equality, because $e \in W$. We may also ask that W be symmetric about e , in the sense that

$$(41.6) \quad W^{-1} = W,$$

since otherwise we can replace W with $W \cap W^{-1}$.

If W is any subset of G , then let W^n be the subset of G consisting of products of n elements of W for each positive integer n , so that

$$(41.7) \quad W^{n+1} = W^n W = W W^n$$

for every n . If $e \in W$ and W satisfies (41.6), then it is easy to see that

$$(41.8) \quad \widehat{W} = \bigcup_{n=1}^{\infty} W^n$$

is a subgroup of G . If W is also an open set in G , then \widehat{W} is an open subgroup of G . If W satisfies (41.4) for some $E \subseteq G$, then we have that

$$(41.9) \quad E W^n \subseteq E$$

for every $n \geq 1$, and hence

$$(41.10) \quad E \widehat{W} \subseteq E.$$

Similarly, (41.5) implies that

$$(41.11) \quad \widehat{W} E \subseteq E.$$

This shows that if E is uniformly separated from its complement with respect to the left uniformity on G , then E is A_L -separated from its complement for some open subgroup A of G , where A_L is as in (35.1). Similarly, if E is uniformly separated from its complement with respect to the right uniformity on G , then E is A_R -separated from its complement for some open subgroup A of G , where A_R is as in (35.2). In both cases, if we also have that $e \in E$, then we get that $A \subseteq E$. If G is strongly 0-dimensional at e with respect to the left or right uniformity, as in Section 33, then it follows that there is a local base for the topology of G at e consisting of open subgroups.

If A is any open subgroup of G , then it is easy to see that A is A_L and A_R -separated from its complement in G . If there is a local base for the topology of G at e consisting of open subgroups, then it follows that G is strongly 0-dimensional at e with respect to the left and right uniformities. This implies that G is strongly 0-dimensional at every point with respect to the left and right uniformities, using invariance under translations. More precisely, G is uniformly 0-dimensional with respect to the left and right uniformities under these conditions, as in Section 39.

Alternatively, if E is a subset of G , then let U_E be the subset of $G \times G$ that corresponds to the equivalence relation whose equivalence classes are E and its complement. As in Section 27, E is uniformly separated from its complement with respect to the left uniformity on G if and only if U_E is an element of the left uniformity on G . In this case, the discussion in the previous section implies that there is an open subgroup A of G such that A_L is contained in U_E . It follows that E is A_L -separated from its complement, as before. Similarly, if E is uniformly separated from its complement with respect to the right uniformity, then one can use an analogous argument to get an open subgroup A of G such that E is A_R -separated from its complement.

42 Open subgroups

Let G be a group, and let \mathcal{A} be a nonempty collection of subgroups of G . Suppose that \mathcal{A} is compatible with conjugations on G , in the sense that for each $A \in \mathcal{A}$ and $g \in G$ there is a $B \in \mathcal{A}$ such that

$$(42.1) \quad B \subseteq g A g^{-1}.$$

In particular, this holds when \mathcal{A} is invariant under conjugations on G , in the sense that

$$(42.2) \quad g A g^{-1} \in \mathcal{A}$$

for every $A \in \mathcal{A}$ and $g \in G$. Of course, if the elements of \mathcal{A} are normal subgroups of G , then this condition holds automatically. If G is a topological group, then the collection of all open subgroups of G has this property, because of continuity of translations.

Let $\tau_{\mathcal{A}}$ be the collection of subsets W of G such that for each $x \in W$ there are finitely many elements A_1, \dots, A_n of \mathcal{A} with the property that

$$(42.3) \quad x \left(\bigcap_{j=1}^n A_j \right) \subseteq W.$$

This is equivalent to saying that for each $x \in W$ there are finitely many elements A_1, \dots, A_n of \mathcal{A} such that

$$(42.4) \quad \left(\bigcap_{j=1}^n A_j \right) x \subseteq W,$$

because \mathcal{A} is supposed to be compatible with conjugations, as in the preceding paragraph. Let us say that \mathcal{A} behaves well with respect to finite intersections if the intersection of finitely many elements of \mathcal{A} always contains another element of \mathcal{A} as a subset. In this case, it suffices to take $n = 1$ in (42.3) and (42.4). If G is a topological group, then the collection of all open subgroups of G is closed under finite intersections.

It is easy to see that $\tau_{\mathcal{A}}$ defines a topology on G . Note that

$$(42.5) \quad \mathcal{A} \subseteq \tau_{\mathcal{A}},$$

because the elements of \mathcal{A} are subgroups of G . By construction, \mathcal{A} is a local sub-base for $\tau_{\mathcal{A}}$ at e . If \mathcal{A} behaves well with respect to finite intersections, then \mathcal{A} is a local base for $\tau_{\mathcal{A}}$ at e . One can also check that G is a topological group with respect to $\tau_{\mathcal{A}}$. More precisely, continuity of translations on G with respect to $\tau_{\mathcal{A}}$ is built into the definition of $\tau_{\mathcal{A}}$. Similarly, continuity of the group operations on G at e with respect to $\tau_{\mathcal{A}}$ follows from the fact that the elements of \mathcal{A} are subgroups of G .

If G satisfies the first or 0th separation condition with respect to $\tau_{\mathcal{A}}$, then

$$(42.6) \quad \bigcap_{A \in \mathcal{A}} A = \{e\},$$

which means that for each $x \in G$ with $x \neq e$ there is an $A \in \mathcal{A}$ such that $x \notin A$. Conversely, (42.6) implies that G is Hausdorff with respect to $\tau_{\mathcal{A}}$. As in Section 39, G is automatically uniformly 0-dimensional with respect to the left and right uniformities associated to $\tau_{\mathcal{A}}$. If (42.6) holds, then it follows that G is uniformly totally separated with respect to the left and right uniformities associated to $\tau_{\mathcal{A}}$, as in Section 33. One can also check this more directly, using the fact that each subgroup A of G is A_L and A_R -separated from its complement, as in the previous section. This implies that A is uniformly separated from its complement with respect to the left and right uniformities associated to $\tau_{\mathcal{A}}$ when $A \in \mathcal{A}$. Of course, any uniform space which is strongly totally separated is totally separated as a topological space, and hence Hausdorff as a topological space.

Suppose that G is already a topological group with respect to some topology τ , and let us take \mathcal{A} to be the collection of all open subgroups of G with respect to τ . Thus \mathcal{A} is invariant under conjugations and closed under finite intersections, as before. In this situation,

$$(42.7) \quad \tau_{\mathcal{A}} \subseteq \tau,$$

because \mathcal{A} is contained in τ by hypothesis. In particular, this implies that open subgroups of G with respect to $\tau_{\mathcal{A}}$ are open with respect to τ . Conversely, every open subgroup of G with respect to τ is in \mathcal{A} and hence in $\tau_{\mathcal{A}}$, by construction.

Suppose for the moment that G is strongly totally separated with respect to the left or right uniformity associated to τ , as in Section 33. This implies that for each $x \in G$ with $x \neq e$ there is a subset E of G such that $e \in E$, $x \notin E$, and E is strongly separated from its complement with respect to the left or right uniformity on G associated to τ , as appropriate. As in the previous section, it follows that there is an open subgroup A of G with respect to τ such that E is A_L or A_R -separated from its complement, as appropriate. Note that $A \subseteq E$ under these conditions, because $e \in E$, as before. Thus $x \notin A$, while $e \in A$ automatically, so that (42.6) holds with this choice of \mathcal{A} .

Conversely, if (42.6) holds with this choice of \mathcal{A} , then we have already seen that G is strongly totally separated with respect to the left and right uniformities associated to $\tau_{\mathcal{A}}$. This implies that G is strongly totally separated with respect to the left and right uniformities associated to τ , because of (42.7).

43 Commutative groups

Let A be a commutative group, with the group operations expressed additively. Thus left and right translations on A are the same. If $d(x, y)$ is a q -semimetric on A for some $q > 0$, then $d(x, y)$ is invariant under translations on A when

$$(43.1) \quad d(x + a, y + a) = d(x, y)$$

for every $a, x, y \in A$. In this case, we have that

$$(43.2) \quad d(-x, -y) = d(x, y)$$

for every $x, y \in A$, using also the symmetry condition (1.2). If $d(x, y)$ is a translation-invariant semi-ultrametric on A , then open and closed balls in A with respect to $d(x, y)$ centered at 0 are subgroups of A , as before.

If A is a commutative topological group, then the corresponding left and right uniformities on A are the same. One can check that A is a commutative topological group with respect to the topology determined by any nonempty collection \mathcal{M} of translation-invariant q -semimetrics on A . In this case, the uniformity on A just mentioned is the same as the one determined by \mathcal{M} .

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$, as in Section 22. Thus

$$(43.3) \quad d(x, y) = |x - y|$$

defines a q -metric on k , which is obviously invariant under translations on k as a commutative group with respect to addition. This leads to a topology on k in the usual way, and it is easy to see that k is a topological group with respect to addition and this topology. If $|\cdot|$ is an ultrametric absolute value function on k , then (43.3) is an ultrametric on k , and in particular k is uniformly 0-dimensional with respect to the corresponding uniformity. In this case, the open and closed balls in k of positive radius with respect to (43.3) are open subgroups of k with respect to addition.

If n is a positive integer, then let $n \cdot 1$ be the sum of n 1's in k , where 1 refers to the multiplicative identity element in k . If $|\cdot|$ is an ultrametric absolute value function on k , then it is easy to see that

$$(43.4) \quad |n \cdot 1| \leq 1$$

for every $n \geq 1$, using (22.7). A q -absolute value function $|\cdot|$ on k is said to be *archimedian* if there are positive integers n such that $|n \cdot 1|$ is arbitrarily large. It suffices to have

$$(43.5) \quad |n_0 \cdot 1| > 1$$

for some positive integer n_0 , because

$$(43.6) \quad |n_0^j \cdot 1| = |(n_0 \cdot 1)^j| = |n_0 \cdot 1|^j$$

for every positive integer j . Otherwise, $|\cdot|$ is said to be *non-archimedian* on k if there is a finite upper bound for $|n \cdot 1|$ over all positive integers n , which implies that (43.4) holds for every $n \geq 1$, by the preceding argument.

Thus ultrametric absolute value functions are non-archimedian, and it is well known that the converse also holds. If $|\cdot|$ is an archimedian q -absolute value function on k , then it follows in particular that k has characteristic 0. This leads to a natural embedding of the field \mathbf{Q} of rational numbers into k , and $|\cdot|$ induces an archimedian q -absolute value function on \mathbf{Q} . In this case, a famous theorem of Ostrowski implies that the induced q -absolute value function on \mathbf{Q} is the same as a positive power of the standard absolute value function on \mathbf{Q} . Another famous theorem of Ostrowski implies that k can be identified

with a sub-field of the field \mathbf{C} of complex numbers, using a positive power of the standard absolute value function on \mathbf{C} .

If p is a prime number, then it is well known that the p -adic absolute value $|x|_p$ defines an ultrametric absolute value function on \mathbf{Q} . The p -adic numbers are obtained by completing \mathbf{Q} with respect to the corresponding p -adic metric.

Let V be a vector space over a field k , and let N be a q -seminorm on V with respect to a q -absolute value function $|\cdot|$ on k for some $q > 0$. Thus

$$(43.7) \quad d(v, w) = N(v - w)$$

defines a q -semimetric on V which is invariant under translations on V as a commutative group with respect to addition. If \mathcal{N} is a nonempty collection of q -seminorms on V , then we get a collection \mathcal{M} of translation-invariant q -semimetrics on V in this way.

References

- [1] J. Cassels, *Local Fields*, Cambridge University Press, 1986.
- [2] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics **242**, Springer-Verlag, 1971.
- [3] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bulletin of the American Mathematical Society **83** (1977), 569–645.
- [4] G. David and S. Semmes, *Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure*, Oxford University Press, 1997.
- [5] G. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.
- [6] G. Folland, *Real Analysis*, 2nd edition, 2nd edition, Wiley, 1999.
- [7] F. Gouvêa, *p -Adic Numbers: An Introduction*, 2nd edition, Springer-Verlag, 1997.
- [8] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.
- [9] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Volumes I, II, Springer-Verlag, 1970, 1979.
- [10] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1975.
- [11] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1969.
- [12] J. Kelley, *General Topology*, Springer-Verlag, 1975.

- [13] J. Kelley, I. Namioka, et al., *Linear Topological Spaces*, Springer-Verlag, 1976.
- [14] J. Kelley and T. Srinivasan, *Measure and Integral*, Springer-Verlag, 1988.
- [15] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.
- [16] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.
- [17] W. Rudin, *Fourier Analysis on Groups*, Wiley, 1990.
- [18] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.
- [19] H. Schaefer and M. Wolff, *Topological Vector Spaces*, 2nd edition, Springer-Verlag, 1999.
- [20] L. Steen and J. Seebach, *Counterexamples in Topology*, 2nd edition, Dover, 1995.
- [21] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [22] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, with the assistance of T. Murphy, Princeton University Press, 1993.
- [23] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.
- [24] M. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, 1975.
- [25] F. Trèves, *Topological Vector Spaces, Distributions, and Kernels*, Dover, 2006.